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# BIDIMENSIONAL LIQUID BRIDGES IN A GRAVITY FIELD

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**Abstract**—The analytical solution for the different shapes that a bidimensional liquid bridge in a gravity field may adopt is reviewed, criticizing the well-known limit of Heywang. The work is shown in the context of a wider-scope program to model the fluid-mechanical behaviour of molten bridges in the floating zone and related techniques of crystal growth, which has blossomed with the advent of microgravity platforms.

## 1. INTRODUCTION

The purest man-made materials today are used in the semiconductors industry and are obtained by the crucible-free solidification from a molten bridge held in vacuum or an inert gaseous ambient by surface tension forces at the liquid–gas interface. Several techniques[1] have been developed that mainly differ in the geometry of supports that bound the molten bridge, which itself is axisymmetric or planar, in most cases.

The physico-chemical and thermo-fluid-dynamical phenomena involved in these processes are so entangled that partition of research tasks is required. If one concentrates on the simple fluid-mechanical problem of determining the possible shapes a liquid-bridge may adopt and its stability, there is still a large field of investigations to deal with a variety of geometries, contact conditions, and applied force fields. The best known configuration is the near-cylindrical liquid bridge anchored to equal solid discs, where the effect of disc separation, liquid volume, axial rotation (used in practice to uniformice the temperature field) and axial gravity, have been quantified (see Martinez *et al.*[2] for a recent review).

The effect of gravity on the shape and stability of liquid drops with different support conditions is well known[3], but for liquid bridges there has been no closed formulation. The pioneering work of Heywang in 1956[4] only dealt with liquid bridges with one cylindrical end condition, what he postulated to model the solidification front in a floating zone process. Besides this assumption being inaccurate in general (the growing angle being 11° instead of 0° for silicon[5], for instance) we find a mistake in the derivation of his most celebrated result: the maximum height for a molten bridge (following his model) is 2.75 times the capillary length instead of the value 2.84 he found.

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These investigations are integrated in a major program of research in the fluid-mechanical modeling of floating zones, where a large experimental program is pursued in real and simulated microgravity conditions[6].

## 2. FORMULATION

With the nomenclature and configuration introduced in Fig. 1, Laplace–Young equation of capillary pressure reads:

$$\sigma \frac{\dfrac{d^2X}{dZ^2}}{\left[1+\left(\dfrac{dX}{dZ}\right)^2\right]^{3/2}} + (p|_{Z=0} - p_{atm}) - \Delta\rho g Z = 0 \quad \text{with } 0 \leqslant Z \leqslant L \quad (1)$$

where $\sigma$, $\Delta\rho$, $g$ and $p_{atm}$ are known (surface tension, liquid–gas density jump, gravity and ambient pressure) and $p|_{Z=0}$ (the liquid pressure at $Z = 0$) may be known if the liquid bridge is fed with the help of a leveled reservoir, or, more often, it is unknown if the liquid is fed with a syringe or naturally by melting a solid zone (then, the volume is assumed constant).

The geometry is scaled with the capillary length, $\sqrt{\sigma/(\Delta\rho g)}$, and the origin of heights is chosen at a level where the internal and external pressures should match, i.e.:

$$x \equiv \frac{X}{\sqrt{\dfrac{\sigma}{\Delta\rho g}}} \quad z \equiv \frac{Z}{\sqrt{\dfrac{\sigma}{\Delta\rho g}}} - \frac{p|_{Z=0} - p_{atm}}{\sqrt{\sigma\Delta\rho g}}. \quad (2)$$

Equation (1) then takes the form

$$\frac{\dfrac{d^2x}{dz^2}}{\left[1+\left(\dfrac{dx}{dz}\right)^2\right]^{3/2}} - z = 0 \quad (3)$$

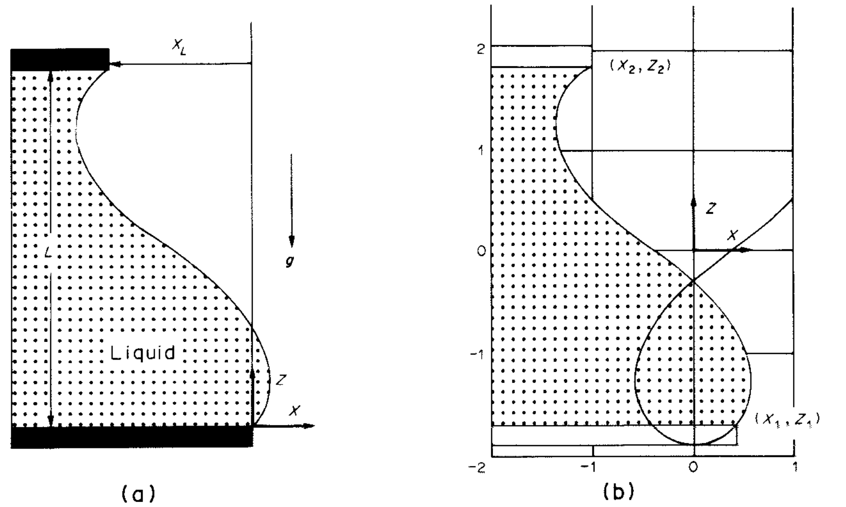


Fig. 1. Sketch of the (bidimensional) configuration analyzed. (a) Dimensional variables. (b) non-dimensional variables [eqn (2)], centered at a height where the pressure in the liquid should equal that of the atmosphere.

which, introducing the local slope of the shape (Fig. 1), $\gamma$,

$$\gamma \equiv \text{arc tg}\,\frac{dz}{dx} \tag{4}$$

and integrating once (3) yields

$$\cos\gamma - 1 - \frac{z^2 - z_0^2}{2} = 0 \tag{5}$$

that furnish a simple phase diagram as plotted in Fig. 2, that helps a lot in locating the different limiting shapes that may arise. From (4), one obtains

$$\int dx = \int \frac{dz}{\text{tg}\gamma} = \int \frac{\dfrac{\sin\gamma\, d\gamma}{\sqrt{z_0^2 - 1 + \cos\gamma}}}{\text{tg}\gamma}$$

$$= \int \frac{\cos\gamma\, d\gamma}{\sqrt{z_0^2 - 1 + \cos\gamma}} \tag{6}$$

which can be expressed[7] in terms of elliptic integrals of the first and second class, $F$ and $E$, as follows:

$$F(\phi, \alpha) = \int_0^{\phi} \frac{d\phi}{\sqrt{1 - \sin^2\alpha \sin^2\phi}},$$

$$E(\phi, \alpha) = \int_0^{\phi} \sqrt{1 - \sin^2\alpha \sin^2\phi}\, d\phi. \tag{7}$$

For $|z_0| < 2$

$0 \leqslant |z| \leqslant |z_0|$

$$r = -z_0[E(\phi, \alpha) - F(\phi, \alpha)] - \frac{2}{z_0} F(\phi, \alpha)$$

with

$$\alpha = \text{arc sin}\,\frac{2}{|z_0|}$$

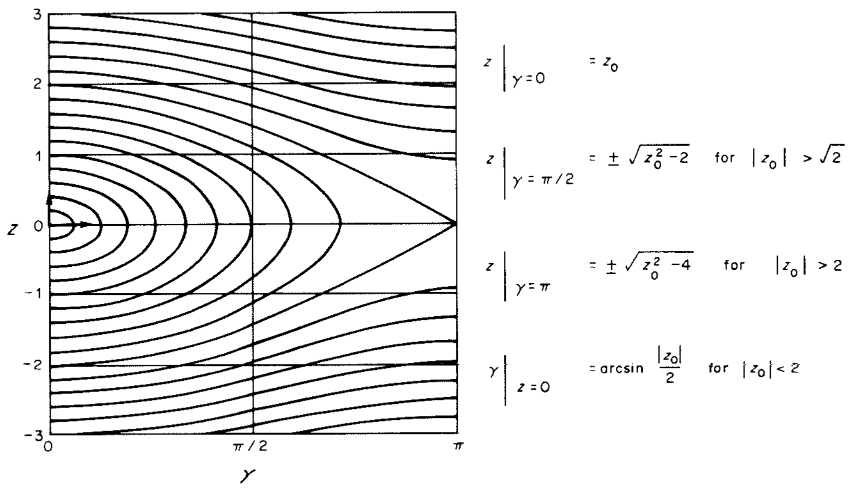


Fig. 2. Phase diagram (vertical coordinate vs slope) [eqn (5)], corresponding to half a period of the generic curves [eqn (6)] a piece of which represents the actual shape of the liquid bridge.

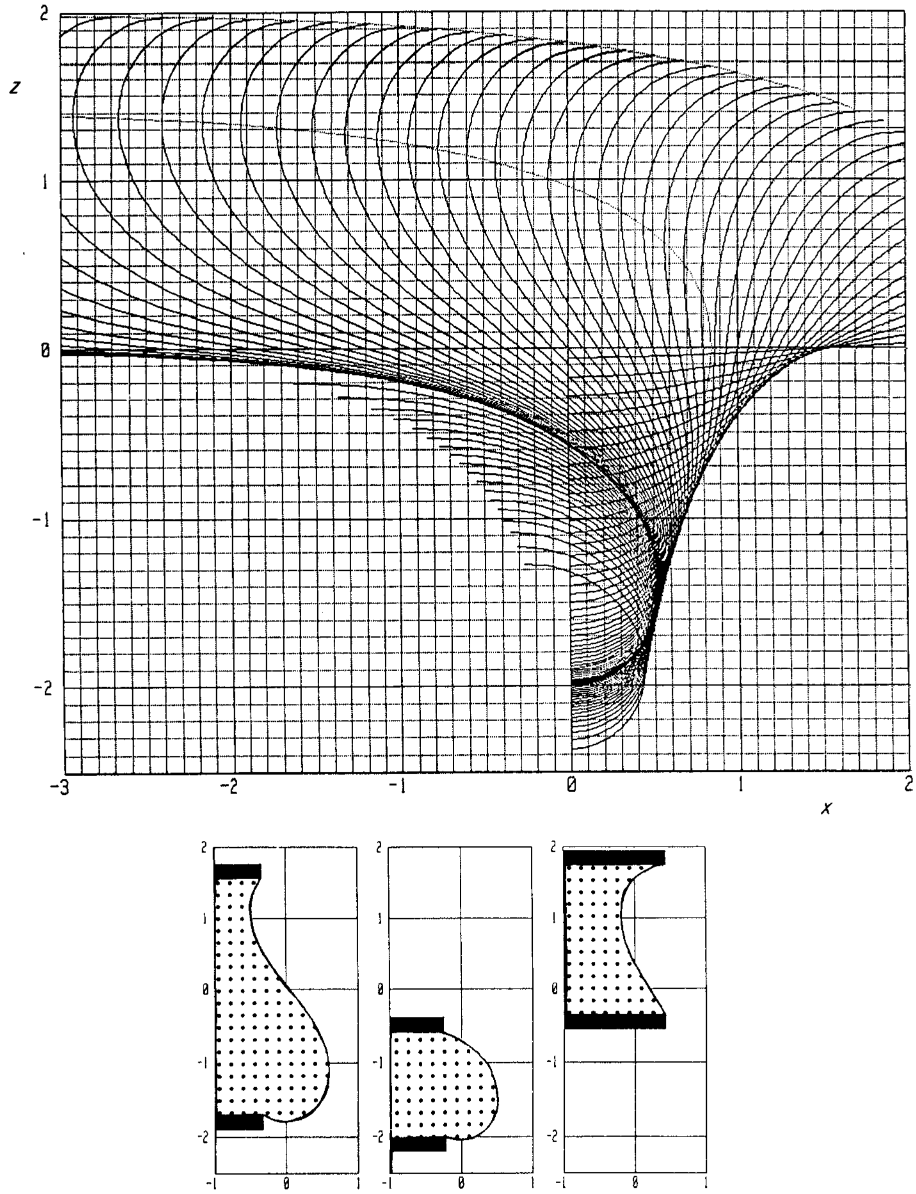


Fig. 3. Half-periods of the family of curves from which all bidimensional shapes can be realized. Curves transpassing the vertical origin show an inflexion there (no capillary pressure there). All curves are horizontally periodic. The three samples illustrate how liquid bridges may look like in the case of equal supports.

and

$$\phi = \frac{\gamma}{2} = \arcsin \frac{\sqrt{z_0^2 - z^2}}{2}. \tag{8}$$

For $|z_0| = 2$

$$0 < |z| \leqslant 2$$

$$r = \mathrm{SGN}(z)\left[\sqrt{4 - z^2} - \arg\mathrm{Ch}\frac{2}{|z|}\right]. \tag{9}$$

For $|z_0| > 2$

$$\sqrt{z_0^2 - 4} \leqslant |z| \leqslant |z_0|$$

$$r = 2E(\phi, \alpha) - F(\phi, \alpha)$$

with

$$\alpha = \arcsin \frac{|z_0|}{2} \quad \text{and} \quad \phi = \arccos \frac{z}{z_0}. \tag{10}$$

Figure 3 shows a general view of the family of curves and three examples that better illustrate how they apply to actual liquid bridges. Notice that only flat solid supports (and thus no re-entry liquid shapes) are considered. Besides, no limiting contact angle condition is imposed, although this will further limit the attainable shapes.

### 3. RESULTS

Heywang[4] considered the special case of liquid bridges with vertical slope ($\gamma = \pi/2$) in one of the discs, and for equal supports he found a maximum bridge height of

$$\frac{L}{\sqrt{\dfrac{\sigma}{\Delta\rho g}}} = z_2 - z_1 = 2.84, \text{ with } z_1 = -1.07 \quad (11)$$

that has been widely published in the literature.

We think he got a mistake in the derivation {from his eqns (23) and (24) in[14]}, because from his eqn (23) we got $z_1 = -1.71$ and $z_2 - z_1 = 2.67$. But, even more, this is not the maximum height for his model of molten zone between equal end diameters (large) and with $\gamma = \pi/2$ at the upper, that yields a maximum $z_2 - z_1 = 2.75$ for slightly hanging bulge, as clearly seen in Fig. 4, where the family of shapes with $\gamma = \pi/2$ in either the lower or upper end is considered (upper $\gamma$ values are limited to $\gamma \geqslant 0$ assuming a flat solid end).

Even if unequal end diameters (large) were considered, the maximum height would still be limited to $2\sqrt{2} = 2.83$ capillary lengths.

However, if the $\gamma = \pi/2$ constraint is released, one finds a maximum value of $z_2 - z_1 = 3.64$, corresponding to an anti-symmetric shape with $\gamma = 0$ at both ends. This value has already appeared in the literature[8, 9] with different degrees of accuracy: Coriell and colleagues found a value of 3.6 both, numerically and experimentally, and Langbein quotes a 3.6693. The exact value is obtained from eqn (8) when $r|_{z=0} = 0$, i.e. solving $E(\phi, \alpha) - (1 - 2/z_0^2)F(\phi, \alpha) = 0$ with $\phi = \arcsin(|z_0|/2)$, $\alpha = \arcsin(2/|z_0|)$ and $z_2 - z_1 = 2|z_0|$. It is also interesting to note that the latter shape, when only the upper half

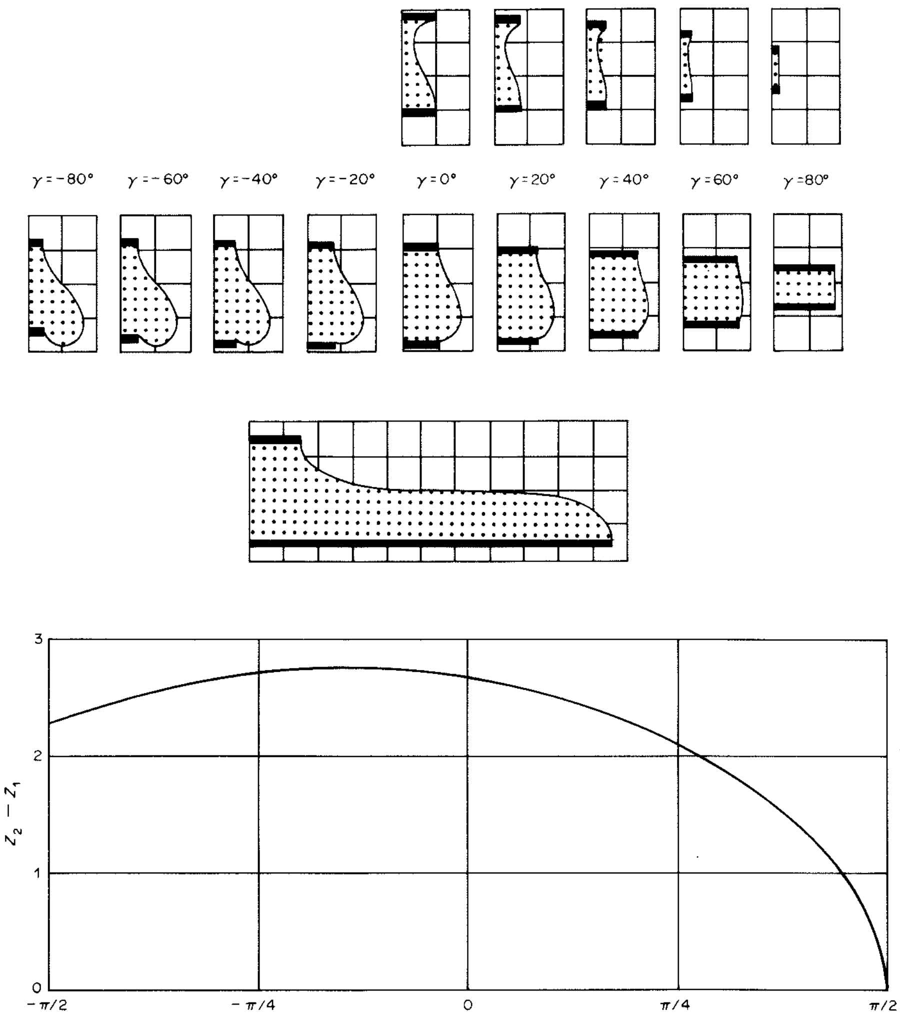


Fig. 4. Family of shapes with vertical slope at one of the two equal supports, and a special case with widely different supports (half of it would correspond to a meniscus between one support and an unbound medium).

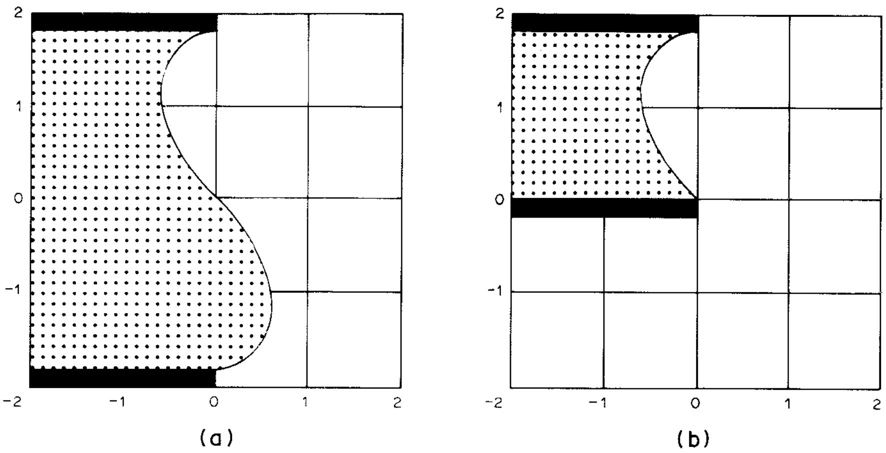


Fig. 5. (a) Highest bidimensional bridge between equal supports (if unequal ends were considered, the outmost height would be 4 capillary lengths instead of 3.64). (b) Lowest liquid-edge angle attainable between equal supports.

is considered, happens to have the smallest liquid-edge angle, 49°, that may be realized at the bottom of two equal supports (see Fig. 5).

#### 4. CONCLUSIONS

The problem considered here applies to real molten shapes in the containerless processing of materials, either because the diameter of axisymmetric configurations is much larger than the meniscus height, or because a ribbon configuration is actually used.

The main result to point out is that the maximum height values published in the literature for this type of meniscus seem to be inaccurate, and exact values, with the proper formulation, are here presented.

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