1.6. SPACELAB EXPERIMENT

FLOATING ZONE HYDROSTATICS

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An analytical solution of the equilibrium shapes and the static stability limits of liquid bridges held by surface tension between two coaxial disks is given following the variational approach. The effect of rotation on the stability of a cylindrical zone is considered in more detail.

1. INTRODUCTION

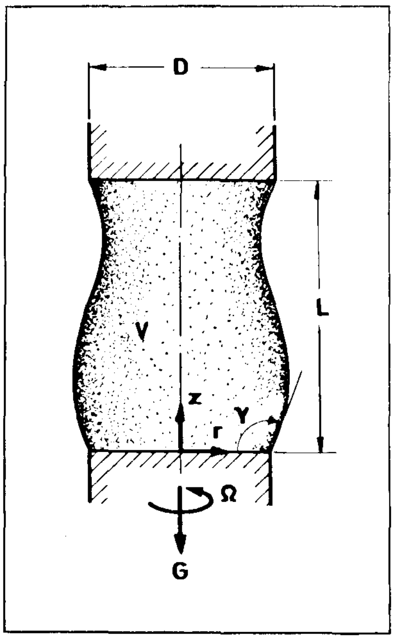
Floating liquid zones appear in a variety of applications: crystal growth, electrophoresis, surface chemistry, etc. The most developed one is the controlled solidification of semiconductor materials from a containerless molten mass. Although it is widely used on Earth under normal gravity conditions (Czochralski's technique), as a result of the hydrostatic pressure the zone can not be higher than few milimeters; this severe constraint is released in absence of gravity and much bigger and perfect zones can be obtained. In this papel the equilibrium configurations and the static stability limits are established under this new conditions.

The history of floating zones goes up to the beautiful experiments of Plateau, who developed the well known neutral buoyancy technique (also named after him) of balancing the hydrostatic pressure in the zone with that of a surrounding immiscible liquid of equal density.

A closely related study to that of the equilibrium shapes has been done by Orr et al. 1975 concerning pendular rings, but the possible equilibrium shapes were already given by Plateau 1973 and Howe 1887 in terms of the nodoide and unduloide figures. The solution for the axisymmetric shapes are summarized in the Appendix.

The stability limit of cylindrical zones coincides with that of cylindrical jets found by Rayleigh 1896. Other stability studies such as Gillis 1967, Mason 17970 and Wues 1976 deal also with cylindrical zones. For non-cylindrical zones we present a general static stability limit analysis. Results are shown present a general static stability limit analysis. Results are shown in

Fig. 1. Coordinates and parameters of the floating zone. Most of the results presented here concerns with zones for $G=\Omega=O$, for which the only stable equilibrium shapes are axisymmetric with a plane of symmetry midway between the disks.

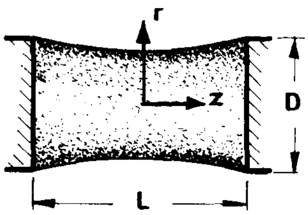
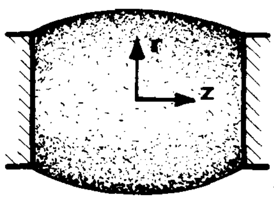
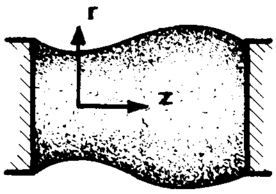


terms of two non-dimensional parameters, $L/D = \Lambda$ and $V/D^3 = v$, characterizing the geometry of the support (disks of diameter D separated a distance L) and the liquid volume V (see figure 1).

2. EQUILIBRIUM AND STABILITY OF FLOATING ZONES

We pose the problem of finding how the equilibrium shape of the free surface changes with L for a constant L (and D, of course). We also look for the range of L which enables a stable configuration of the liquid joining the disks. A similar problem is that of keeping constant L and consider the change in shape as the volume V is varied; in general, we are interested on finding the change in shape and the bounds (for disruption not to occur) when following a certain evolution $f(\Lambda, v) = 0$. Before going into the details, let the solution be anticipated as shown in

Table A-1. Parametric solution for axisymmetric shapes.

			
Comments	$v \leq \pi\Lambda/4$	$v \geq \pi\Lambda/4$	always unstable
$r(\varphi,\varphi^*,\alpha)$ =	$\frac{1}{2}\frac{A(\varphi,\alpha)}{A(\varphi^*,\alpha)}$	$\frac{1}{2}\frac{A(\varphi,\alpha)}{A(\varphi^*,\alpha)}$	$\frac{1}{2}\frac{A(\varphi,\alpha)}{A(\varphi^*,\alpha)}$
$z(\varphi,\varphi^*,\alpha)$ =	$\frac{\Lambda}{2}\left\lvert\frac{B(\pi/2,\alpha)-B(\varphi,\alpha)}{B(\pi/2,\alpha)-B(\varphi^*,\alpha)}\right\rvert$	$\frac{\Lambda}{2}\frac{B(\varphi,\alpha)}{B(\varphi^*,\alpha)}$	$\frac{\Lambda}{2}\frac{B(\varphi,\alpha)}{B(\pi/2,\alpha)}$
with φ in the range	$\varphi^* \leq \varphi \leq \pi-\varphi^*$	$\varphi^* \leq \varphi \leq \varphi^*$	$\varphi^* \leq \varphi \leq \pi+\varphi^*$
$\Lambda(\varphi^*,\alpha)$ =	$\left\lvert\frac{B(\pi/2,\alpha)-B(\varphi^*,\alpha)}{A(\varphi^*,\alpha)}\right\rvert$	$\left\lvert\frac{B(\varphi^*,\alpha)}{A(\varphi^*,\alpha)}\right\rvert$	$\left\lvert\frac{B(\pi/2,\alpha)}{A(\varphi^*,\alpha)}\right\rvert$
$v(\varphi^*,\alpha)$ =	$\left\lvert\frac{C(\pi/2,\alpha)-C(\varphi^*,\alpha)}{A^3(\varphi^*,\alpha)}\right\rvert$	$\left\lvert\frac{C(\varphi^*,\alpha)}{A^3(\varphi^*,\alpha)}\right\rvert$	$\Lambda^3\left\lvert\frac{C(\pi/2,\alpha)}{B^3(\pi/2,\alpha)}\right\rvert$
$s(\varphi^*,\alpha)$ =	$\left\lvert\frac{D(\pi/2,\alpha)-D(\varphi^*,\alpha)}{A^2(\varphi^*,\alpha)}\right\rvert$	$\left\lvert\frac{D(\varphi^*,\alpha)}{A^2(\varphi^*,\alpha)}\right\rvert$	$\Lambda^3\left\lvert\frac{D(\pi/2,\alpha)}{B^2(\pi/2,\alpha)}\right\rvert$
$p(\varphi^*,\alpha)$ =	$4A(\varphi^*,\alpha)/(1+\cos\alpha)$	$4A(\varphi^*,\alpha)/(1+\cos\alpha)$	$4A(\varphi^*,\alpha)/(1+\cos\alpha)$
m =	$1/\cos\alpha$	$\cos\alpha$	$1/\cos\alpha$

Functions A, B, C and D are defined by:

$A(\varphi,\alpha) = (1-\sin^2\alpha\sin^2)^{1/2}$

$B(\varphi,\alpha) = \cos\alpha\, F(\varphi,\alpha) + E(\varphi,\alpha)$

$C(\varphi,\alpha) = \pi[\sin^2\alpha\sin\varphi\cos\varphi A(\varphi,\alpha) - \cos\alpha\, B(\varphi,\alpha) + 2(1+\cos\alpha)^2 E(\varphi,\alpha)]/12$

$D(\varphi,\alpha) = \pi(1+\cos\alpha)\, E(\varphi,\alpha)$

where F and E are the standard elliptic integrals of the first and second class (its series expansion form is recommended for calculating).

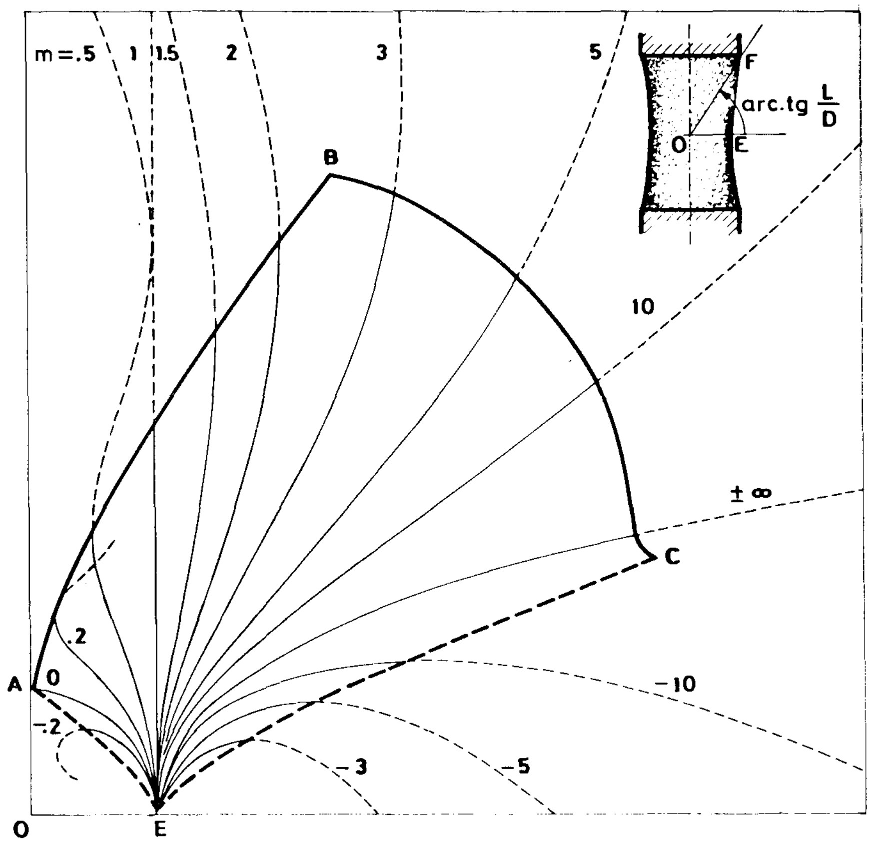


Fig. 2. Stable equilibrium shapes of a floating zone held between disks of diameter D placed a distance L apart, for $G = \Omega = 0$. Curves m = constant correspond to the outline EF shown in the insert (line EF must be drawn for each particular value of L/D). The successive cuts of the line EF with curves m = constant give the actual position of the zone edge, F. Points F inside ABCE correspond to stable configurations (and unstable ones if outside).

figures 2 and 3. Some comments on the assumptions made are needed: first of all we assume a perfect wetting liquid anchored at the edges of the disks where any contact angle, Υ, can take place; there exist an upper bound for Υ based only in stability criteria but it is so big (bigger than π) that the liquid is expected to overflow in any real experiment (due to other effects not accounted in this model also, when withdrawning a zone with $\Upsilon = 0$ it will detach from the edges of the disk. Thence it seems reasonable to limit our study to instabilities taking place for $0 \leq \Upsilon \leq \pi$.

The variational approach for solving the hydrostatic problem is based on minimizing the total free energy of the zone (Landau and Lifshitz 1959, p. 232); the internal free energy of an incomprensible fluid depends only on the volume of the fluid and not on the shape of its surface. The latter affects, firstly the surface free energy $\sigma \int dS$ and secondly, the energy in the external field (gravity and centrifugation) which is $\rho \int G Z\, dV - \frac{1}{2} \rho \Omega^2 \int r^2\, dV$. Thus the hydrostatic problem can be written

$$\sigma \int_S dS + \rho G \int_V Z\, dV - \frac{1}{2} \rho\, \Omega^2 \int_V r^2\, dV = \text{minimum}. \qquad (1)$$

This minimum is to be determined subject to the condition

$$\int_v dv = \text{constant}, \qquad (2)$$

which expresses the fact that the volume of the fluid is known.

Equations (1) and (2) with the condition of attachment at the disk edge, furnish the variational formulation of the problem. A function Φ is built in the way

$$\Phi = \sigma \int dS\, \rho G \int Z\, dV - \frac{1}{2} \rho\, \Omega^2 \int r^2\, dV + \lambda \int dV, \qquad (3)$$

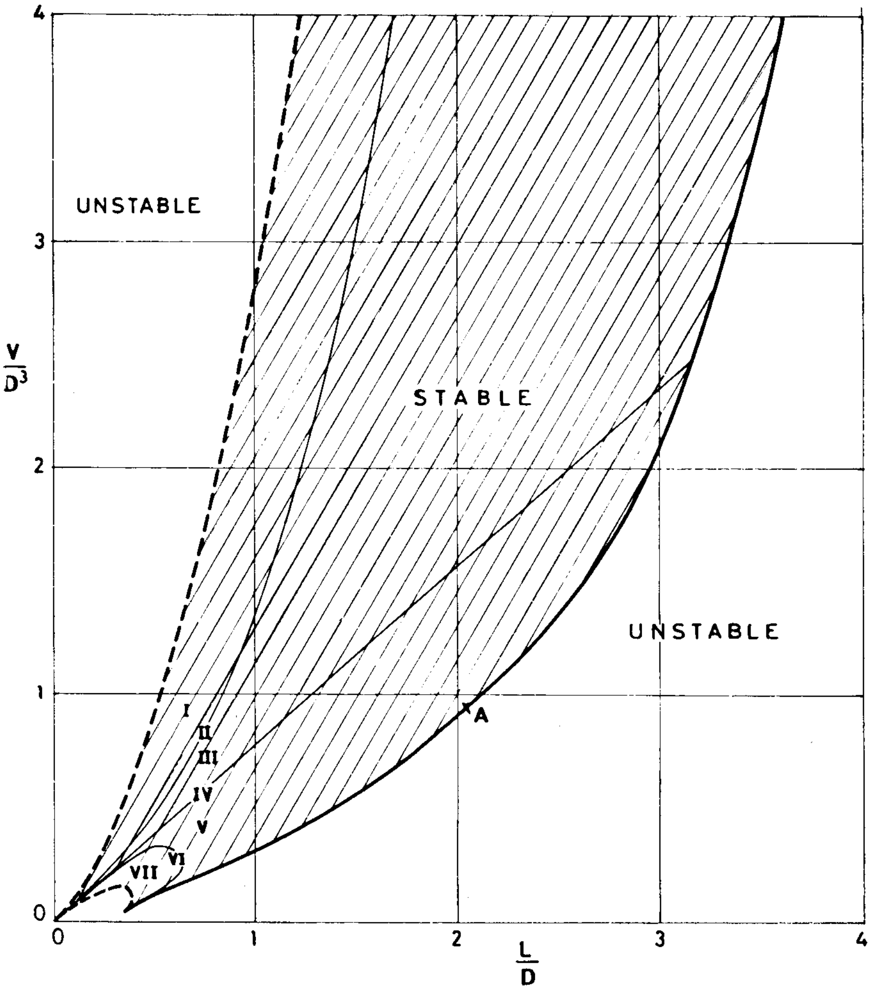
where λ is a Lagrange multiplier and standard variational theory give the equilibrium states

$$\delta\Phi = 0, \qquad (4)$$

the stability limit

$$\delta^2\Phi = 0, \qquad (5)$$

Fig. 3. Stability limits of a floating zone held between disks of diameter D placed a distance L apart, for $G=\Omega=0$. For very short zones ($L/D < \cdot 5$, say) the successive shapes when liquid is being removed are: I convex nodoid, II spherical segment, III convex unduloid, IV cylinder, V concave unduloid, VI catenoid and VII concave nodoid. For more slender zones ($\cdot 6 < L/D < \pi$) the last two figures are never seen because the zone breaks with a concave unduloid shapes (the breakage is symmetric for $L/D < 2$, till point A, and non-symmetric for $L/D > 2$). For greater slenderness, even the cylindrical zone is no longer obtainable.



and the stable configurations

$$\delta^2\Phi > 0. \qquad (6)$$

It is worth noticing that applying the Lagrangian method in the same way as for mechanical systems yields the same formulation.

Limiting to axisymmetric shapes greatly reduces the mathematical treatment (nearly axisymmetric shapes will be considered when dealing with the effect of rotation). Let the non-dimensional variables and functions be defined as

$$\frac{r}{D} = r \quad \frac{Z}{D} = z \quad \frac{L}{D} = \Lambda \quad \frac{\rho G D^2}{\sigma} = g$$

$$\frac{\rho\Omega^2 D^3}{2\sigma} = \omega^2 \quad \frac{V}{D^3} = v \quad \frac{S}{D^2} = s \quad \frac{\Phi}{\sigma D^2} = \Phi \qquad (7)$$

Thus equation (3) once integrated in θ and r becomes

$$\Phi = \int_0^\Lambda f(z, r, r')\, dz = \int_0^\Lambda 2\pi \left(r\sqrt{1 + r'^2} + \frac{g}{2} r^2 z - \frac{\omega^2}{4} r^4 - \frac{pr^2}{2}\right) dz \qquad (8)$$

where $p = -\frac{\lambda D}{\sigma}$ is a non-dimensional Lagrange multiplier which happens to be the non-dimensional pressure-difference between the origin and the outer atmosphere. Symbol r represents, from now on, the surface equation as r(z), primes denoting derivatives respect to z.

To satisfy condition (4), Euler equation

$$\frac{\partial f}{\partial r} - \frac{d}{dz}\left(\frac{\partial f}{\partial r'}\right) = 0 \qquad (9)$$

must be verified. From equations (8) and (9) we get

$$\frac{r''}{(1+r'^2)^{3/2}} - \frac{1}{r(1+r'^2)^{1/2}} - gz + \omega^2 r^2 + p = 0 \qquad (10)$$

which represents the local equilibrium condition at the free surface. For $g = \omega = 0$ equation (10) admits a closed solution in terms of elliptic integrals; the results are presented in figure 2 (details are given in the Appendix).

Let $N(r)$ and $\bar{r}(z)$ represent equation (10) and a known solution of it for $g = \omega = 0$, respectively. Any solution of (10) in the neigbourhood of $\bar{r}(z)$ can be expanded as

$$r(z) = \bar{r}(z) + \varepsilon\eta(z) + 0(\varepsilon^2), \qquad \varepsilon << 1, \quad (11)$$

$\eta(z)$ satisfying the so called Jacobi equation

$$\frac{d}{dz}\left(\frac{\partial^2 f}{\partial \bar{r}'^2}\eta'\right) + \eta\frac{d}{dz}\left(\frac{\partial^2 f}{\partial\bar{r}\partial\bar{r}'} - \frac{\partial^2 f}{\partial\bar{r}^2}\right) + \left[\frac{\partial^2 f}{\partial\bar{r}\partial p} - \frac{d}{dx}\left(\frac{\partial^2 f}{\partial\bar{r}'\partial p}\right)\right]p_1 = 0 \qquad (12)$$

which is deduced from (5) or sustituting (11) in (10), $\bar{p}$ being the pressure in the origin of zone $\bar{r}$, and p_1 its variation with ε; equation (12) can be also looked as the first term in the expansion of $N(r)$, namely

$$N(r) = N(\bar{r}) + \varepsilon L(\eta) + 0(\varepsilon^2) \equiv 0 \qquad (13)$$

Solving (12) with homogeneous conditions yields the static stability limits (curve ABC in figure 2).

In fact, expansion (13) provides a useful procedure for approaching equation (10) for g and ω small since if solution r for $g = \omega = 0$ is known, the first approximation of $r(z, g, \omega)$ is $r = \bar{r} + \varepsilon\eta$ with η satisfying the non-homogeneous equation $L(\eta) = -N(\bar{r})/\varepsilon$.

3. THE EFFECT OF ROTATION ON THE STABILITY OF A CYLINDRICAL ZONE

This section intends to show the simplest case of the hydrostatic problem but in a more general approach mainly concerning the possibility of non-axisymmetric equilibrium shapes.

A solution satisfying (4) when $g = 0$, $v = \pi\Lambda/4$, any Λ and any ω, is the cylindrical shape $r = 1/2$; any other solution (if it exists) in its neigbourhood can be written as

$$r(\theta, z) = 1/2 + \varepsilon\eta(\theta, z) + 0(\varepsilon^2), \qquad \varepsilon << 1, \quad (14)$$

Introducting (14) in (13), using the non-dimensional variables defined in (7) and linearising yields

$$\phi = \iint\left[\frac{1}{2} - \frac{\omega^2}{64} - \frac{p}{8} + \varepsilon\eta\left(1 - \frac{\omega^2}{8} - \frac{p}{2}\right) - \varepsilon^2\eta^2\left(\frac{\eta_{\theta\theta}}{\eta} + \frac{\eta_{zz}}{4\eta} + \frac{3\omega^2}{8} + \frac{p}{2}\right)\right] d\theta dz. \qquad (15)$$

Equilibrium requires $\delta\Phi = 0$, thus

$$1 - \frac{\omega^2}{8} - \frac{p}{2} = 0. \qquad (16)$$

The stability limit will be given by $\delta^2\Phi = 0$, thus

$$-\left(\frac{\eta_{\theta\theta}}{\eta} + \frac{\eta_{zz}}{4\eta} + \frac{3\omega^2}{8} + \frac{p}{2}\right) = 0. \qquad (17)$$

According to the linearization one can develop $\eta(\theta, z)$ in Fourier series and analyse a general single mode, namely

$$\eta(\theta, z) = A\cos m\theta \sin\frac{\pi n z}{\Lambda}. \qquad (18)$$

The requirement of volume constancy imposes one o fthe following conditions:

$$\left.\begin{array}{llll} \text{if} & m = 0 & \text{then} & n \ \text{odd} \\ \text{if} & m \neq 0 & \text{then} & n \ \text{even} \end{array}\right\} . \qquad (19)$$

Substituting (18) in (17) and taking into account (16) yields

$$m^2 + \frac{n^2\pi^2}{4\Lambda^2} - \frac{\omega^2}{4} - 1 = 0. \qquad (20)$$

The first modes of instability are

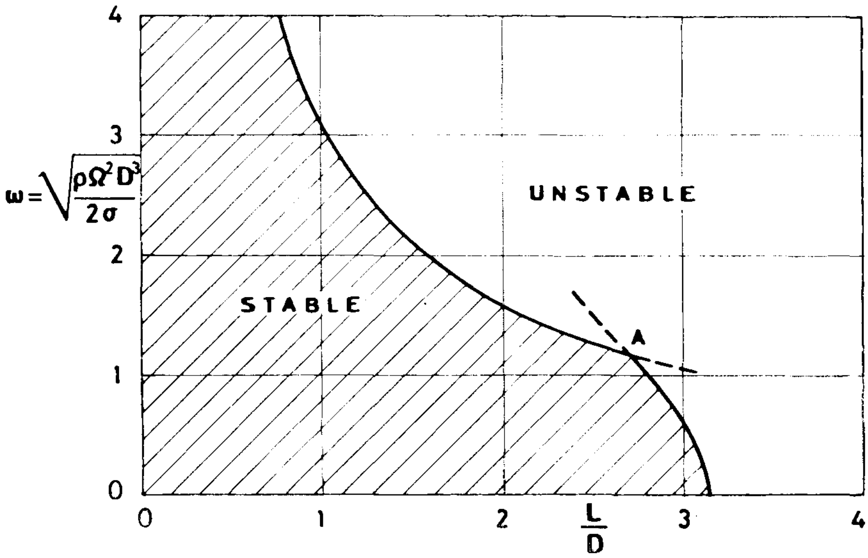
$$\left.\begin{array}{ll} \text{for } m = 0 \text{ and } n = 2 & \dfrac{\pi^2}{\Lambda^2} - \dfrac{\omega^2}{4} - 1 = 0 \\ \text{for } m = 1 \text{ and } n = 1 & \dfrac{\pi^2}{\Lambda^2} - \omega^2 = 0 \end{array}\right\} (21)$$

Now it is easy to discern between stable and unstable shapes with condition (6). Figure 4 shows these results.

CONCLUSIONS

An analytical solution of the equilibrium shapes and the static stability limits of liquid bridges held by surface tension between two coaxial disks has been presented following a variational approach. The more relevant remarks are the following:

Fig. 4. Reduction on the stability limit of cylindrical zone due to a spin rate at constant angular velocity Ω. For Ω larger than that corresponding to point A a new mode of breakage appear: the skipping rope or C mode G=O.



Filling or removal of liquid in short zones ($L/D < .4$) does not present special features, but for slender zones a minimum volume of liquid is attained after which the liquid bridge disrupts for $.4 < L/D < 2$ the breakage is symmetric (with a contact angle $\gamma < \pi/2$), leading to two equal spherical caps one on each disk; finally, for $L/D > 2$ the breakage is non-symmetric (always taking place with $\gamma = \pi/2$) leading to different size caps. Figure 5 gives a complete picture of the solution, allowing a quick estimation of any evolution with floating zones.

The effect of solid body rotation is important in two respects: first, it reduces the stability limits, and second, it provides a new mode of breakage, the skipping rope or C mode, by means of which, the liquid would leave the supports. This new instability can only occur for large spin rates ($\Omega > \sqrt{\frac{8\sigma}{3\rho D^3}}$).

The effect of microgravity is equivalent to that of the disalignement in a buckling problem.

It appears that the study both of the dynamic stability and of the breakage process of a floating zone are promising areas of further research.

ACKNOWLEDGEMENT

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APPENDIX

For $g = \omega = 0$ the equilibrium problem is that of finding the function $r(z, \Lambda, v)$ which represents the free surface (of revolution) of a liquid volume $V/D^3 = v$ held between two equal parallel-coaxial disks with aspect ratio $L/D = \Lambda$, satisfying equation (10) with $r(0) = s(\Lambda) = 1/2$. Table A-1 gives a summary of the solution in parametric form. Under the same assumptions, equation (12), that gives the stability limits, becomes

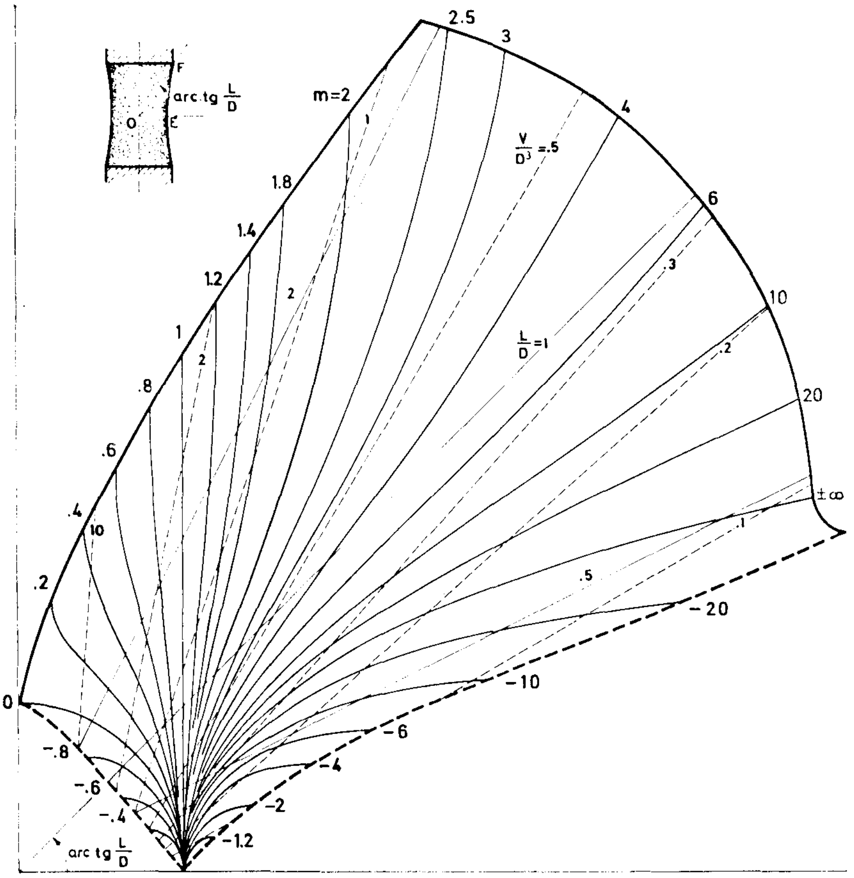
$$\eta'' + \frac{\bar r' + \bar r'^2 - 3\,\bar r\,\bar r'\,\bar r''}{\bar r\,(1 + \bar r'^2)}\,\eta' + \frac{1 + \bar r'^2}{\bar r^2} + (1 + \bar r'^2)^{3/2} p_1 = 0,$$

which can easely be integrated, once the general solution of (10) is known as a function of two parameters; in effect, two independent solutions for the homogeneous are supplied by the respective partial derivatives of it. Although this procedure seems tedious it is not difficult to show that at last we arrive at a simple necessary condition for bifurcation to occur, namely

$$J(\varphi^*, \alpha) = \begin{vmatrix} \partial\Lambda/\partial\varphi^* & \partial\Lambda/\partial\alpha \\ \partial v/\partial\varphi^* & \partial v/\partial\alpha \end{vmatrix} = 0$$

which yields the stability limits (curve ABC in figure 2). Additional details may be found in Martínez 1978.

Fig. 5. Summary-diagram for the hydrostatics of a floating liquid zone. Any stable evolution can be easely analyzed as, for instance the successive shapes for a constant volume V when separating the disks, or the successive shapes for a constant separation L when filling or removal the liquid. $\Lambda = L/D$, $v = V/D^3$ and m is a parameter defining the meridian curve.



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