FLOATING ZONE UNDER REDUCED GRAVITY—AXISYMMETRIC EQUILIBRIUM SHAPES

I. Martinez Herranz
ETSIA Aerodynamics Laboratory, University of Madrid, Spain

ABSTRACT

The mathematical formulation of the problem of calculating the axisymmetric equilibrium shapes of an incompressible fluid mass, held by surface-tension forces between two parallel coaxial discs, and rotating about its axis, is considered. The effect of gravity is neglected. Both variational methods, based on extreme kinematic potential, and curvature methods, based on local equilibrium conditions, are used to calculate the form of the equilibrium surface. The stability of the zone to axisymmetric perturbation is considered in detail.

1. INTRODUCTION

The classical theory of equilibrium shapes of isolated fluids held together by surface tension can be found in the work of Poincaré, Rayleigh and many others. The contribution of the present work is to consider appropriate boundary conditions where liquid, solid, and outer gas are in contact.

The problem to be treated is the equilibrium configuration of a rotating liquid zone bounded by a gas and two end discs. The bifurcation points, defining the limits above which the axisymmetric equilibrium configuration becomes unstable, are analysed in a cylindrical-shaped zone. The influence on these results of the boundary conditions at the end discs is discussed. It is shown that the widely quoted, largest value of the stable length in the absence of rotation,

$$L_{max} = 2\,\pi\,R$$

is only valid in our case when the disc edges anchor the liquid zone. It is no longer valid when the edge contact is free to move, in which case

$$L_{max} = \pi\,R$$

2. LIST OF SYMBOLS

A = constant in Equation (6)
B = constant in Equation (11)
C = constant in Equation (11)
L = zone length, taken as unit length, also Lagrangian of a system
P = pressure
R = radius of a cylindrical zone
T = kinetic energy made dimensionless with σL^2
U = potential energy made dimensionless with σL^2
V = volume of the floating zone made dimensionless with L^3
a = disc radius made dimensionless with L
n = branching mode

p = dimensionless pressure difference from zone axis to atmosphere

$$p=\frac{P_o-P_a}{\sigma}L$$

r = dimensionless radius of curvature
x = dimensionless longitudinal co-ordinate
y = dimensionless transversal co-ordinate
Ω = angular velocity
α = dimensionless wave number, $\alpha=2\pi L/\lambda$
θ = contact angle
λ = wavelength
ρ = liquid density
σ = liquid/gas surface-tension coefficient
ω = dimensionless rotation parameter

$$\omega=\frac{\rho\Omega^2L^3}{2\sigma}$$

Superscripts

'

Accents indicate derivatives with respect to x.

Subscripts

a = atmospheric
m = meridian
q = parametric differentiation
r = revolution
0 = conditions at disc $x=0$, although P_0 indicates pressure at the liquid zone axis
1 = conditions at disc $x=1$.

3. MATHEMATICAL FORMULATION

The assumptions on which the present study is based are the following:

(i) a liquid zone in solid rotation is bounded by a gas and two end discs
(ii) the properties of the interface (taken as an infinitely thin zone) are uniform
(iii) contact angle is constant
(iv) dynamic effects are not taken into account.

The zone configuration is shown in Figure 1a, while Figure 1b sketches its geometry, indicating the relevant parameters.

There are two general ways of formulating this problem mathematically; namely, the variational approach and the curvature approach.

3.1 VARIATIONAL METHOD

As in every problem of classical dynamics, we can write down the Lagrangian of the system, expressing its kinematic potential [1]

$$L=T-U \tag{1}$$

with

$$T=\frac{\pi}{2}\omega\int_0^1 y^4\,\mathrm{d}x$$

and

$$U=2\pi\int_0^1 y\sqrt{1+y'^2}\,\mathrm{d}x \tag{2}$$

Now, we look for an extremal of the Lagrangian integral

$$\int_0^1 L\mathrm{d}x\equiv\text{extremal} \tag{3}$$

subject to the restriction that the zone volume

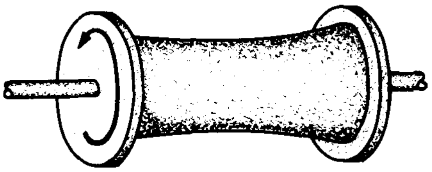
$$V=\pi\int_0^1 y^2\,\mathrm{d}x \tag{4}$$

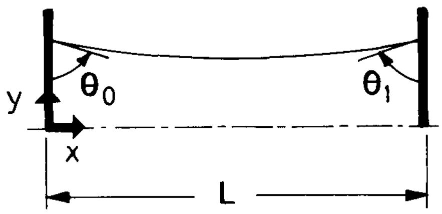
is known beforehand.

3.2 CURVATURE METHOD

The differential equation governing the shape of the zone is obtained by expressing the local equilibrium of the liquid surface, so that the local mean curvature is proportional to the pressure drop. Figure 2 is a schematic for the mean curvature calculation [2].

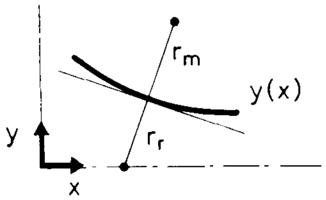
The variational or the curvature approach yields the following second-order, nonlinear, differential equation:





(a) *(b)*

Figure 1. Sketch of the zone.



$$\frac{1}{r_m} = \frac{y''}{(1+y'^2)^{3/2}}$$

$$\frac{1}{r_r} = \frac{-1}{y(1+y'^2)^{1/2}}$$

Figure 2. Mean-curvature calculus.

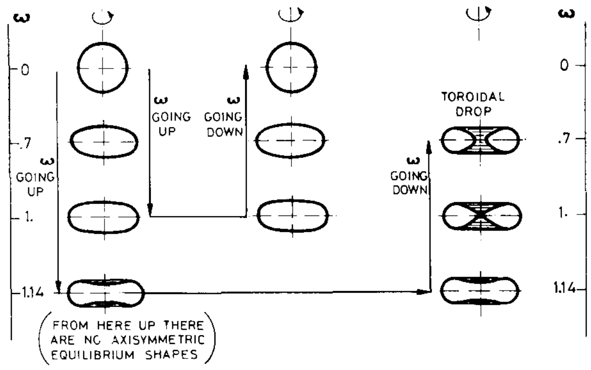


Figure 3. Axisymmetric equilibrium shapes of isolated rotating fluid masses.

$$\frac{y''}{(1+y'^2)^{3/2}} - \frac{1}{y(1+y'^2)^{1/2}} + \omega y^2 + p = 0 \qquad (5)$$

Equation (5) can be integrated once to give

$$\frac{1}{\sqrt{1+y'^2}} = \pm \left[\frac{\omega}{4} y^3 + \frac{p}{2} y + \frac{A}{y}\right] \qquad (6)$$

Further analytical progress requires the use of hyperelliptical integrals and a resorting to numerical computations. However, if the free surface meets the axis (broken zone), the A/y term in Equation (6) disappears and the problem becomes somewhat easier. This particular problem has been treated by Poincaré [3] (1902), Rayleigh [4] (1896), Lichtenstein [5] (1933), Landau [6] (1959), and many others [7]. The equilibrium shapes in that case are summarised graphically in Figure 3 because it could be of great interest for the study of the broken zone.

4. PECULIARITIES OF THE FLOATING ZONE

In the above-mentioned case of an isolated droplet, the sphere was the only equilibrium shape in the absence of rotation. In our floating zone case, instead of one, there are infinite possible shapes depending upon the contact at the end discs. Some of them are well known: cylinders, spherical portions, catenoids, and so on. Hence the problem of finding the equilibrium shapes, even in the absence of rotation, becomes as hard as in the general case of rotation.

Figure 4 shows sketches of the possible types of contact conditions:

(a) Contact angle preserved. In this case, the disc is large enough to allow the free motion of the liquid boundary on it, $y'(0) = -\cot \theta_0$.

(b) The liquid boundary is anchored at the disc edge, $y(0) = a_0$.

In what follows, we will limit ourselves to the study of nearly cylindrical shapes, in order to obtain, as easily as possible, several results concerning the position of the bifurcation points.

For example, if we wish to follow an analytical approach to find the solutions for nearly cylindrical

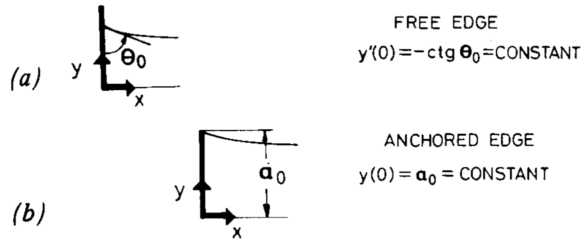


Figure 4. Contact conditions.

shapes, we can develop y in power series of x in Equation (6), and then integrate. As a first approximation, we get, for equal contact angles at both end discs,

$$y=y_0-2x(1-x)\cot\theta \tag{7}$$

for $80^\circ \leqslant \theta \leqslant 100^\circ$. This expression agrees with numerical computations within a relative error of less than 1%. θ could be either the real contact angle or the angle at the disc edge (Fig. 5). It should be remarked that the influence of rotation does not appear in the above solution.

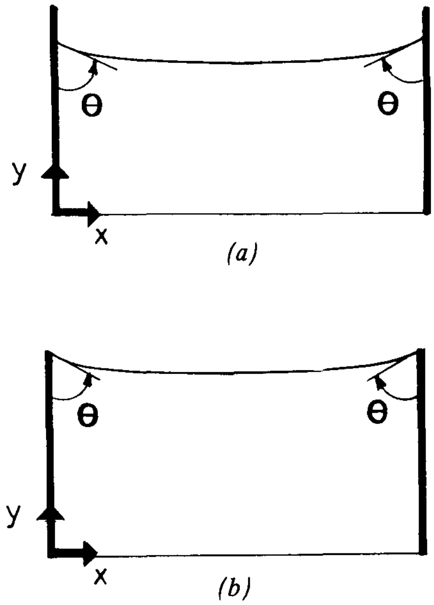


Figure 5. Limits of validity for Equation (7).

5. BIFURCATION POINTS

We now wish to know whether the cylindrical shape is the only solution to our problem, when $\theta_0=\theta_1=\frac{\pi}{2}$, or whether there are some other solutions that also satisfy the same differential equation (5) and boundary conditions required by the problem.

Let us characterise these solutions by a parameter q. Differentiating with repect to it, the formulation for the perturbation becomes

$$y_q''+\left[3y'(1+y'^2)^{1/2}(\omega y^2+p)-\frac{2y'}{y}\right]y_q'+\left[\frac{1+y'^2}{y^2}+2\omega y(1+y'^2)^{3/2}\right]y_q=0 \tag{8}$$

$$\int_0^1 y_q\, y\, \mathrm{d}x=0 \tag{9}$$

with homogeneous boundary conditions.

If the undisturbed shape is a circular cylinder of radius R, Equation (8) becomes

$$y_q''+\left(\frac{L^2}{R^2}+2\omega\frac{R}{L}\right)y_q=0 \tag{10}$$

the general solution of which is

$$y_q=B\cos\alpha\, x+C\sin\alpha\, x \tag{11}$$

where α, the wave number, is given by

$$\alpha^2=\frac{L^2}{R^2}+2\omega\frac{R}{L} \tag{12}$$

Adding the appropriate boundary conditions, we find the following three cases:

Edge Case

$y_q(0) \quad = 0\rightarrow B=0$

$y_q(1) \quad = 0\rightarrow$ either $C=0$ (trivial solution) or $\alpha=\pi n$

$\int_0^1 y_q\,\mathrm{d}x \quad = 0\rightarrow \alpha=2\pi n$

which indicates that the smallest value of α is $\alpha=2\pi$. Going back to dimensional quantities,

$$L_{max}=2\pi n R\left[\sqrt{1+\frac{\rho\Omega^2R^3}{\sigma}}\right]^{-\frac{1}{2}}$$

This result has already been obtained by Gillis [8]. The case $\Omega = 0$ corresponds to the well-known result by Rayleigh. Experimental evidence supporting this result has been given by Plateau [9] in 1873 and by Mason [10] in 1970, both in the case $\Omega = 0$. The case $\Omega \neq 0$ has been checked experimentally by Carruthers & Grasso [11] in 1972. Neutral buoyancy conditions were used in all these experiments.

Slipping Case

$y_q'(0) = 0 \rightarrow C = 0$

$y_q'(1) = 0 \rightarrow$ either $B = 0$ (trivial solution) or $\alpha = \pi n$

$\int_0^1 y_q \, dx = 0 \rightarrow \alpha = \pi n$

Hence, the smallest value of α is $\alpha = \pi$. Now, in terms of dimensional variables,

$$L_{max} = \pi R \sqrt{1 + \frac{\rho \Omega^2 R^3}{\sigma}}$$

This result, not found in the available literature, shows that floating zones which are free to move on the end discs are less stable than those anchored to the edge of the end plates.

Mixed Conditions

$y_q(0) = 0 \qquad y_q'(1) = 0$

In this case there are no nontrivial solutions.

6. GENERAL COMMENTS AND CONCLUSIONS

To sum up the present work, the following comments of a general nature should be made:

(i) Exact knowledge of the figures of equilibrium of axisymmetric floating zones under reduced gravity conditions requires machine computing, even in the simple cases of either no rotation or solid rotation. Only a few instances are amenable to either exact or approximate analytical treatment. A general parametric tabulation of the numerical results is being prepared.

(ii) The shape of nearly cylindrical zones in solid rotation is remarkably insensitive to angular speed.

(iii) Most previous analyses of floating-zone stability concern zones anchored to the borders of the end discs. While this particular boundary condition is significant in the study of phase-changing floating zones, other conditions would be more realistic in several instances. In particular, it is shown that the floating zone sliding on both end discs is more unstable than that anchored to the discs, in the sense that its maximum stable length is much shorter.

(iv) No experimental results concerning these sliding floating zones have been found in the literature, and it is therefore suggested that simple experiments on this topic should be undertaken.

7. ACKNOWLEDGEMENT

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