PUSH AND RELEASE OF A PISTON

Statement

Inside a vertical cylinder of 0.01 m^2 cross-section there is 0.01 kg of nitrogen closed by a 5 kg steel piston at the top. Consider the following process: 1) by a suitable force, the piston is gently moved so as to reduce the initial volume of the gas by 10%; 2) the applied force is suddenly removed. One wants to:

a) Sketch of the evolution in a *z-t* (height-time) diagram.
b) Determine the thickness of the piston and its initial height.
c) State of the system at each point and a *p-V* (pressure-volume) process diagram.
d) Energy change between states, and heat and work transferred through the frontier.
e) *T-s* (temperature-entropy) process diagram.
f) Entropy change and entropy generation for every system between consecutive states, and along the whole process.

Dentro de un cilindro vertical de 0,01 m^2 de sección hay 0,01 kg de nitrógeno encerrado con un émbolo superior de 5 kg de acero. Considérese la siguiente evolución: 1) mediante las fuerzas apropiadas se obliga al pistón a reducir lentamente en un 10% el volumen ocupado por el gas, y 2) se libera el anclaje y se permite el libre movimiento del émbolo. Se pide:

a) Esquema de la evolución en un diagrama altura-tiempo.
b) Determinar el espesor del émbolo y su altura inicial.
c) Valores *p-V-T* en los estados de equilibrio considerados.
d) Variación de energía entre los estados antedichos y para todo el ciclo, así como calor y trabajo transferidos entre los sistemas involucrados.
e) Diagrama *T-s* de la evolución.
f) Variación de entropía y generación de entropía para todos los sistemas entre los estados antedichos y para todo el ciclo.

Solution

a) Sketch of the evolution in a *z-t* (height-time) diagram.

Most problems are not as simple as the previous exercises, and the solution is not immediate but requires some development, mixing appropriately comments on notation and assumptions, analytical deductions (formulation) and numerical computations. To better grasp the problem by the person solving it, and to better transmit the information to the persons using it, it is important to start by a sketch of the system and its evolution, introducing also the nomenclature used, as here in Fig. 1.

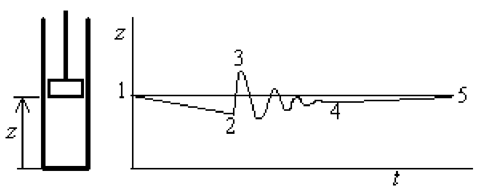

Fig. 1. Time evolution of the piston: 1) initial state, 2) 10% compression, 3) maximum expansion, 4) mechanical equilibrium, 5) thermal equilibrium.

First the initial conditions are studied. From the problem statement we should understand that the piston is initially in mechanical equilibrium so that the pressure in the gas is $p_1=p_0+m_Pg/A$, where p_0 is the ambient atmospheric pressure (assumed as 100 kPa as reasoned in the Introduction), m_P is the mass of the piston, g the gravitational acceleration (assumed as 9.8 m/s^2 as reasoned in the Introduction) and A the cross-section area $A=\pi D^2/4$. In absence of more details, we also take T_0=288 K (15 ºC) as reasoned in the Introduction.

b) Determine the thickness of the piston and its initial height.

The density of steel is assumed available: ρ=7900 kg/m^3 (from Thermal data), so the thickness is $L_P=m_P/(\rho_P A)$=0.064 m.

We obtained the height of the piston from the ideal gas equation of state $pAz=mRT$ with with T_1=288 K and $p_1=p_0+m_Pg/A$=105 kPa, what yields $z_1=mRT/(p_1A)$=0.815 m.

c) State of the system at each point and a p-V (pressure-volume) process diagram.

From 1 to 2 a slow compression takes place, thence enough time for heat transfer may be assumed and the process approximated as isothermal, so that pV=constant, and with $z_2=(V_2/V_1)z_1$, z_2=0.9·0.815=0.734 m, yields $p_2=p_1(V_1/V_2)$=117 kPa with T_2=288 K.

The evolution after release of the piston in state 2 is rather complicated, but the final state of thermodynamic equilibrium (state 5 in Fig. 1) is trivial because temperature and pressure are recovered from the initial state, so that p_5=105 kPa, z_5=0.815 m, with T_5=288 K.

Point 3, the maximum expansion, is obtained assuming that there is no time for heat transfer, and that the friction force F_f between piston and cylinder is small. Upon integration of the momentum equation for the piston:

$$F=m_P\ddot{z}\longrightarrow\int Fdx=\int m_P\ddot{z}dz=\int m_P\ddot{z}\dot{z}dt=\tfrac{1}{2}m_P\dot{z}^2=\int\left(pA-p_0A-m_Pg-F_f\right)dz \quad (1)$$

from state 2 to 3, both with zero speed, neglecting F_f, and with pz^γ=constant, one gets:

$$0=\int_{z_2}^{z_3}p_2\left(\frac{z_2}{z}\right)^\gamma Adz-(p_0A+m_Pg)(z_3-z_2)=p_2z_2^\gamma\left(\frac{z_3^{1-\gamma}-z_2^{1-\gamma}}{1-\gamma}\right)-(p_0A+m_Pg)(z_3-z_2) \quad (2)$$

This is one equation with one unknown z_3 that must be numerically or graphically evaluated since it is not explicitly solved algebraically because of the power γ. To do it by hand, once the known values substituted:

$$0=2910-\frac{1891}{z_3^{0.4}}-1049z_3 \quad (3)$$

it is better to start by the expected solution; if it were a linear oscillation around z_1=0.815 m from z_2=0.734 m, it would reach z_5=0.815+(0.815–0.734)=0.896 m, so this is a good start, but we get for the right-hand-side of (3) a value of –6 (the unit is J), instead of 0; trying now with a close value e.g. 0.85 we get 0.2 instead of 0, and interpolating linearly, the zero would correspond to z_3=0.851 m. More accurately result is z_3=0.852 m; from pV^{γ}=constant one gets p_3=95 kPa, and from the equation of state T_3=271 K. Just to help visualise the problem, Fig. 2 presents the function to be cancelled: $f(z)=2910-1891/z^{0.4}-1049z$.

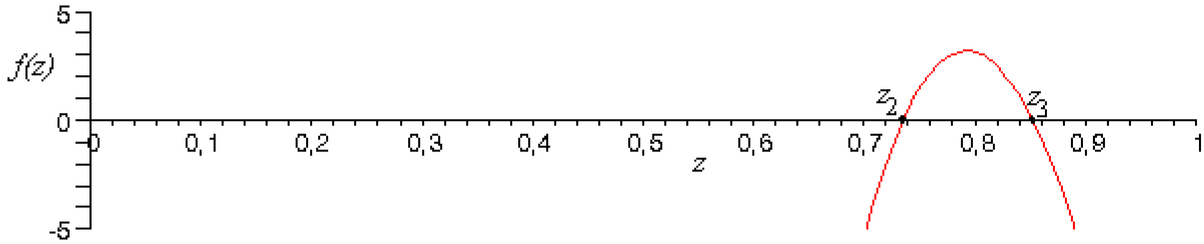

Fig. 2. Plot of the function to be cancelled in (3), i.e. $f(z)=2910-1891/z^{0.4}-1049z$. The trivial solution is $z=z_2$=0.734 m, as in (2), and the one looked for is $z=z_3$=0.852 m.

Point 4 corresponds to the mechanical equilibrium, which is quickly reached even for small-friction systems, far quicker than the thermal equilibrium when gases are participating, that takes a very long time. Because of mechanical equilibrium of the piston, $p_4=p_0+m_Pg/A$=105 kPa, and, assuming that heat transfer is still under-developed, pV^{γ}=constant, because there is not friction inside the system (gas); thence $z_4=z_2(p_2/p_4)^{\gamma}$= $0.734(117/105)^{1.4}$=0.791 m and T_4=279 K.

Point 5 coincides with the initial state, as said before. The *p*-*V*-*T* values (really *z*-*T*-*p* values) are summarised in Table 1, and the evolution in the *p*-*V* diagram shown in Fig. 3.

Table 1. Values of state variables at key points in the evolution of the gas.

state	*z* [m]	*T* [K]	*p* [kPa]
1	0.815	288	105
2	0.734	288	117
3	0.852	271	95
4	0.791	279	105
5	0.815	288	105

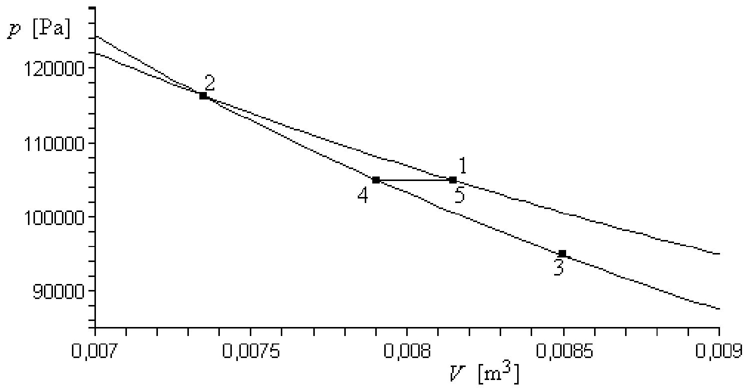

Fig. 3. Evolution in the *p*-*V* diagram. From 1 to 2 isothermal, from 2 to 3 adiabatic oscillations dumped until mechanical equilibrium at point 4, from 4 to 5 isobaric tempering.

d) Energy change between states, and heat and work transferred through the frontier.

The energy change between two states for the trapped gas is $\Delta U=mc_v\Delta T$ where the thermal capacity at constant volume for the gas, c_v, is assumed to be known, usually through the Mayer relation c_p-c_v=R and tabulated c_p values (see Thermal data): $c_v=c_p-R$=1040-297=743 J/(kg·K). The work received by the system (the trapped gas) is computed by $W=-\int p\mathrm{d}V$, once the function $p=p(V)$ is determined by the actual process. The heat received by the system is computed with $Q=\Delta E-W$. The results obtained are summarised in Table 2 and analysed below.

Table 2. Energy change, and work and heat received in the gas system.

process	ΔE [J]	W [J]	Q [J]
1-2	0	90.1	-90.1
2-4	-63.4	-63.4	0
4-5	63.4	-25.3	88.7
1-5	0	1.4	-1.4

Obviously $\Delta E=W+Q$. Also, because of the cyclic process, $\Delta E_{15}=0$, but $W_{15}\neq 0$ and consequently $Q_{15}\neq 0$. Since the work done is the area between the $p=p(V)$ path and the abscissas, the work in the cycle is the area enclosed by the closed path (the little triangle-wise 1-2-4-5 in Fig. 3), that is positive (received by the system) if the cycle is run anti-clockwise, as in this exercise, and negative in the contrary. It will be shown in Chap. 3 that this cycle represents a refrigeration machine, capable of extracting heat (process 4 to 5) from a system below ambient temperature, at the expense of some external work. It is also interesting to note that in the process 1-2, from the 90.1 J received by the gas, only 4.6 J were actually consumed by the external force pushing the piston; the rest were supplied by the decreasing potential energy of the piston, $m_E g\Delta z$=4.0 J, and primarily by the ambient atmosphere, $p_0\Delta V$=81.5 J. From this 4.6 J directly applied, 1.4 J are used for the refrigeration effect (Table 2) and the other 3.2 J dissipated in the friction between piston and cylinder.

e) T-s (temperature-entropy) process diagram.

The generic entropy change between two states for the trapped gas is $S_2-S_1=m(c_p\ln(T_2/T_1)-R\ln(p_2/p_1))$. The numerical values are computed below, but the sketch is easy: an isothermal entropy decrease (heat flowing out), isentropic oscillations, and a final isobaric tempering to re-gain initial conditions.

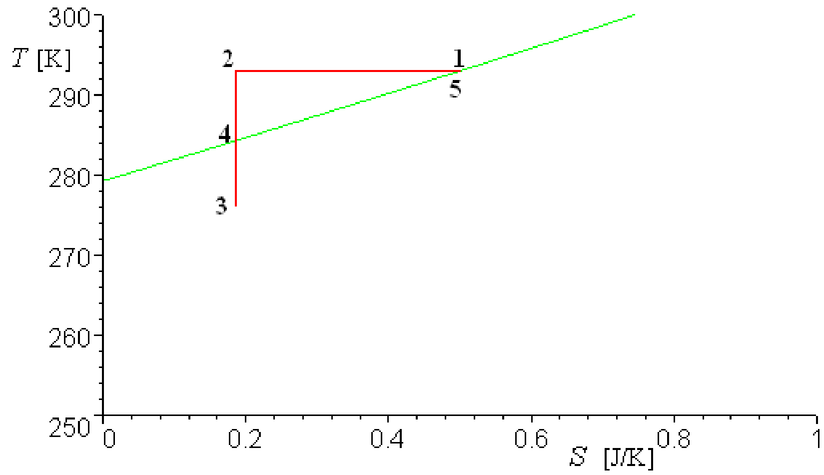

Fig. 4. Evolution in the T-S diagram. Note that the origin of S is arbitrary (here, S_0=0.5 J/K is taken, to centre the process path).

f) Entropy change and entropy generation for every system between consecutive states, and along the whole process.

The generic entropy change between two states for the system, ΔS, has been quoted above, and for the atmosphere it is $\Delta S_{env}=Q_0/T_0$ (with $S_{gen,env}=0$ because it is assumed to be a reversible heat source). But there is a singularity with this modelling; as explained in Exercise 2: we must add a third thermodynamic system, the frontier, to account for all physical processes, since, with the uniform-temperature model for the system and the environment, entropy is generated at the frontier. To skip that singularity, we follow here the trick of extending the environment a little inside the system, to include that dissipation; thence $\Delta S_{env}=Q_0/T$ with $S_{gen,env}>0$. The generation of entropy in the universe is now $S_{gen}=\Delta S+\Delta S_{env}$. The results are presented in Table 3.

From 1 to 2, $\Delta S_{12}=m(c_p\ln(T_2/T_1)-R\ln(p_2/p_1))=0.01(1040\cdot\ln(288/288)-297\cdot\ln(117/105))=-0.313$ J/K. For the environment receives $\Delta S_{env}=Q_0/T_0=90.1/288=0.313$ J/K.

From 2 to 4, gas evolution has been assumed isentropic, $\Delta S=0$, but the entropy of the environment has increased due to friction dissipation. Although there is no time from 2 to 4 for the heated pieces in friction (piston and cylinder walls) to evacuate the energy to the atmosphere (and not to the gas inside), let us assume the model is appropriate, and compute $\Delta S_{env}= Q_0/T_0=-W_{24}/T_0=-63.4/288=-0.222$ J/K.

From 4 to 5, $\Delta S_{45}=m(c_p\ln(T_5/T_4)-R\ln(p_5/p_4))=0.01(1040\cdot\ln(288/279)-297\cdot\ln(105/105))= 0.313$ J/K. For the environment receives $\Delta S_{env}=Q_0/T_0=-88.7/288=0.308$ J/K.

Table 3. Entropy change for the system (trapped gas) and for the environment (including the frontier), and global entropy generation.

	Gas			Environment			Universe		
process	ΔS [J/K]	∫dQ/T [J/K]	S_{gen} [J/K]	ΔS [J/K]	∫dQ/T [J/K]	S_{gen} [J/K]	ΔS [J/K]	∫dQ/T [J/K]	S_{gen} [J/K]
1-2	-0.313	-0.313	0	0.313	0.313	0	0	0	0
2-3-4	0	0	0	0.222	0	0.222	0.222	0	0.222
4-5	0.313	0.313	0	-0.308	-0.313	0.005	0.005	0	0.005
1-2-3-4-5	0	0	0	0.227	0	0.227	0.227	0	0.227

Notice that entropy generation is always positive, both at a single step and for the whole process, whereas entropy changes of a non-isolated system may be positive or negative.

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