



# HEAT AND MASS CONVECTION.

## BOUNDARY LAYER FLOW

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## HEAT AND MASS CONVECTION

We present here some basic modelling of convective process in [Heat and mass transfer](#). [Heat diffusion](#), [mass diffusion](#), and heat radiation are presented separately. Furthermore, mass convection is only treated here as a spin-off of the heat convection analysis that takes the central focus.

### Heat convection: what it is

There cannot be any convected heat, since heat is only defined as thermal-energy flow through an impermeable surface due to a temperature difference across. What we call heat convection is the effect of a fluid flow on heat conduction at a fluid-washed wall; i.e. we intend to apply Newton's law of cooling instead of Fourier's law (see Physical transport phenomena in [Heat and mass transfer](#)):

$$\text{What is heat (flux) convection?} \quad \dot{q} \equiv h(T - T_\infty) = -k\nabla_n T \quad (1)$$

(where  $\nabla_n$  stands for the normal gradient at the wall), aiming at substituting the effect of a real flow field by an empirical boundary condition at the wall, i.e. with the convective coefficient  $h$  in (1) found from global measurements of temperatures and heat fluxes in experiments, instead of by analytically solving the fluid flow (Navier-Stokes' equations) and using (1) to deduce  $h$ . Internal thermal energy (not heat) is convected with the fluid flow, in an amount dependent on a reference energy-level (reference temperature), usually referred to the ambient or sink temperature. When the increase in internal thermal

energy is due to heat transfer at a source, the energy balance for a fluid flow at constant pressure without phase changes and reactions is  $\dot{Q} = \dot{m}c\Delta T$ , what shows that, the same thermal load can be transported by a high mass-flow-rate flow with small temperature jump, or by a low mass-flow-rate flow with high temperature jump, and that thermal-carrier fluids should have high thermal capacity.

Notice that in Fluid Mechanics, there is no Newton's law of cooling, and the only heat-transfer term to be included is Fourier's conduction (and in very special cases thermal radiation emission or absorption through the media).

## Types of heat convection

Heat convection problems may be classified according to:

- Time variation, as steady or unsteady convection. Only a marginal fraction of applications require transient convection analysis (e.g. when the onset of natural convection in a fluid layer heated from below, is studied).
- Flow origin, as forced convection or natural convection. Forced convection occurs when the fluid flow is imposed by other agent than the heat-transfer phenomena under study, i.e. by a pump, a fan, or natural convection from other objects. Natural or free convection occurs when the fluid flow appears as a consequence of the heat-transfer phenomena under study, due to buoyancy forces caused by density gradients in an external force field. Natural convection takes place in all heat convection problems under gravity, but when forced convection is imposed, the latter usually overcomes the former (a combination of the two must be considered at small forcing speeds). Forced convection greatly enhances heat transfer, but demands power consumption. (According to this division, the internal flow in a heat-pipe, due to capillary pumping, is forced, in spite of not consuming external power. Thermo-capillary convection, like Marangoni convection, is also not considered in these notes.)
- Flow regime, as laminar flow or turbulent flow. Turbulent flow is the rule in engineering applications, but laminar flow always exists in some regions, like close to walls and entrance regions. Turbulent convection greatly enhances heat transfer, but increases power consumption too.
- Flow topology, as internal flow or external flow. Internal flow is when we focus on the fluid flowing inside pipes and ducts, whereas external flow is when we focus on the fluid flowing outside pipes and ducts, or around any other solid object. The distinction is not so clear when one considers a portion of a duct (e.g. a flat plate), or open-channel flow, although all these cases are traditionally considered external flow. Some other times, flow topology depends on the detail of the analysis, as in shell-and-tube heat exchanger, when heat transfer can be considered external convection of the shell-fluid around the tubes, or internal convection of a shell-ducted flow to the walls (mainly the internal walls), as for compact plates heat exchangers.
- Flow phases, as single-phase or multi-phase flows. An intermediate type is stratified flow (i.e., homogeneous, heterogeneous: stratified, two bulk phases, and disperse). This division is not only important for permanent multi-phase flow, but for vaporising and condensing flows.

- Flow detail, as detailed heat convection or global heat convection. Most of the times, the empirical approach to convection heat transfer only looks for global values of the convective coefficient around a solid, or along a pipe; but there are cases where temperature variations along the wall must be resolved, either during experiments to compute global  $h$ -values by integration, or during analysis to know if some temperature limit is locally exceeded, and for this purpose a local approach is of interest.
- Flow compressibility is seldom important in heat convection. Flow reactivity, if any, is considered aside as a distributed energy source or sink. Other important fluid flow divisions, like viscous and inviscid flows, or 1D- 2D- and 3D- flows, are of little importance in the study of heat transfer by convection, because of the global empirical approach followed.
- Thermal boundary regulation. Two basic cases are considered: constant wall temperature, and constant heat flux, the former being more closely approached in practice (it is simpler to regulate the temperature of the wall than the heat flux through it, and there temperature is maintained in phase changes of a pure substance), but the latter being simplest to model, since it means a constant source term in the energy balance (and it is the actual case in counter-flow heat exchangers with similar fluid flows). As a matter of fact, both types of control can be advantageously used, as in hot-wire velocimetry (or hot-wire anemometry), where either the wire temperature is controlled (regulating its electrical resistance  $R(T)$  and measuring the required power as a function of flow speed,  $\dot{Q}$ ), or the supplied power to the wire is fixed, and the steady-state temperature difference between wire and fluid measured. The steady-state energy balance,  $\dot{Q} = V^2/R = KA(T_w - T_\infty)$ , allows a calibration against relative flow speed,  $v$ , when a cooling law is assumed (e.g.  $K = a + b\sqrt{v}$ ).

Air is the most ubiquitous fluid in heat convection. All terrestrial animals and most equipment transfer heat to the environment by natural convection in air, with a typical convection coefficient value in the range  $h=5..10$  W/(m<sup>2</sup>·K). For simple cooling/heating load calculations with wind effects, Duffie and Beckmann (1991) rule  $h=a+bv_{wind}$ , with  $a=3$  W/(m<sup>2</sup>·K) and  $b=3$  J/(m<sup>3</sup>·K) may be a first approximation. Notice that the convective coefficient depends on fluid type, flow type, and geometry. For instance, for natural convection from a plate in air, correlations for  $h$  in (1) are (see details in [Forced and natural convection](#))  $h=a(T-T_\infty)^{1/4}$ , with  $a=2.4$  W/(m<sup>2</sup>·K<sup>5/4</sup>) for the upper face of a horizontal surface,  $a=1.3$  W/(m<sup>2</sup>·K<sup>5/4</sup>) for the lower face of a horizontal surface (stable vertical gradient), and  $a=1.8$  W/(m<sup>2</sup>·K<sup>5/4</sup>) for a vertical surface.

## A brief on Fluid Mechanics

Fluid Physics encompasses the nature of fluids (their structure and properties), the fluid forces within and on the boundaries, the transport of mass, momentum and energy, and possible effects of reactive processes, electromagnetic interactions, and so on; Fluid Mechanics concentrates on fluid forces and the transport of momentum. Our interest here is on the transport of energy, but this is linked to the transport of momentum and cannot be studied apart when fluid flow exists; that is why we are presenting below an *ad hoc* summary of the general equations, a rather complicated formulation with the three simple objectives:

- To have an overview of the full equations which are pre-programmed in computational-fluid-dynamic codes (as commercial CFD packages). The user is in charge of selecting the appropriate terms in the equations, and setting the initial and boundary conditions, but the equations are automatically solved.
- To have an idea of the terms retained and the terms neglected in some simple heat-and-mass transfer problems to be analysed in detail, as the boundary-layer flow, and the pipe flow.
- To better understand the rationale for the grouping of dimensional variables into the traditional non-dimensional parameters.

The description of fluid flow makes use of the continuum model, and on the concept of fluid particle, an infinitesimal control system in local thermodynamic equilibrium. Once a reference frame is selected, the motion of fluid particles may be described in two ways:

- Eulerian description, where the unit volume is fixed to the spatial reference frame, and the motion of the particle that at every instant happens to pass by this position, is specified.
- Lagrangian description, where the unit volume moves with the flow relative to the spatial reference frame, and the motion of the same particle at each position, is specified.

Passing from the most-intuitive Lagrangian to the most-used Eulerian description is based on Reynolds transport theorem:

$$\frac{d\Phi_{CM}}{dt} = \frac{d}{dt} \int_{V_{CM}(t)} \phi dV = \int_{V_{CV}(t=t_0)} \frac{\partial \phi}{\partial t} dV + \int_{A_{CV}(t=t_0)} \phi (\vec{v} - \vec{v}_{A_{CV}}) \cdot \vec{n} dA \xrightarrow{V_{CV} \neq f(t)} \int_{V_{CV}} \frac{\partial \phi}{\partial t} dV + \int_{A_{CV}} \phi \vec{v} \cdot \vec{n} dA \quad (2)$$

which says that, for of any conservative property ( $\Phi$  may be mass, momentum, or energy) in a control mass CM (its value being the integral of the specific function over the volume; e.g.  $\phi$  would be mass per unit volume,  $\rho$ ), the variation with time in a permeable system can be computed as the integral of the specific function within the control volume CV, plus the flux of that variable over the permeable area.

Passing from area integrals to volume integral is based on Gauss-Ostrogradski's divergence theorem:

$$\frac{d\Phi_{CM}}{dt} = \int_{V_{CV}} \frac{\partial \phi}{\partial t} dV + \int_{A_{CV}} \phi \vec{v} \cdot \vec{n} dA \stackrel{\iiint_{V_{CV}} \nabla \cdot \vec{v} dV = \iint_{A_{CV}} \vec{v} \cdot \vec{n} dA}{=} \int_{V_{CV}} \left( \frac{\partial \phi}{\partial t} + \nabla \cdot (\phi \vec{v}) \right) dV = 0 \quad (3)$$

which is applied to an infinitesimal volume to get the general equations of Fluid Mechanics: continuity equation, momentum equation, and energy equation.

### Continuity equation

The continuity equation is the mass balance ( $dm_{CM}/dt=0$ ) for a  $dV$  system; with  $\phi=\rho$  we get from (3):

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0, \quad \text{or} \quad \frac{D\rho}{Dt} + \rho \nabla \cdot \vec{v} = 0 \quad (4)$$

where the convective derivative  $D()/Dt \equiv \partial()/\partial t + \vec{v} \cdot \nabla()$  is often introduced to make the writing more compact. In most heat and mass transfer problems, the continuity equation can be reduced to  $\nabla \cdot \vec{v} = 0$  because density changes are usually negligible.

### Momentum equation

The momentum equation is the linear-momentum balance ( $d(m\vec{v})_{CM}/dt = \vec{F} = \vec{F}_V + \vec{F}_A$ ), applied to a  $dV$  system; with  $\phi = \rho\vec{v}$  we get from (3), using the convective derivative:

$$\rho \frac{D\vec{v}}{Dt} = \frac{\partial(\rho\vec{v})}{\partial t} + \nabla \cdot (\rho\vec{v}\vec{v}) = \nabla \cdot \bar{\bar{\tau}} + \rho\vec{g} = -\nabla(p + \rho gz) + \nabla \cdot \bar{\bar{\tau}}, \quad (5)$$

where  $\bar{\bar{\tau}}$  is the stress tensor (such that the force per unit area of normal vector  $\vec{n}$  is  $\vec{f} = \bar{\bar{\tau}} \cdot \vec{n}$ ),  $\vec{g}$  is any volumetric force field (e.g. gravity),  $p$  is fluid pressure (one third of the trace of the stress tensor), and  $\bar{\bar{\tau}}$ ' the viscous component of the stress tensor. In most heat and mass transfer problems (5) can be reduced to the so-called Boussinesq approximation (constant-density flow except for the buoyancy term, proportional to the thermal-expansion coefficient  $\alpha$ , named in honour of the French Academician V. J. Boussinesq, who studied in the late 19th c. convective cooling and turbulence):

$$\frac{D\vec{v}}{Dt} = -\frac{\nabla p}{\rho} - g[1 - \alpha(T - T_0)]\vec{i}_z + \nu \nabla^2 \vec{v} \quad (5a)$$

where  $\nu \equiv \mu/\rho$  is the kinematic viscosity and  $\mu$  the dynamic viscosity.

### Energy equation

The energy equation is the energy balance ( $d(me)_{CM}/dt = \dot{Q} + \dot{W}$ ) for a  $dV$  system; with  $\phi = \rho e$  we get from (3):

$$\rho \frac{De}{Dt} = -\nabla \cdot \vec{q} + \nabla \cdot (\bar{\bar{\tau}} \cdot \vec{v}) \quad (6)$$

which in most heat and mass transfer problems is expressed in terms of temperature:

$$\rho c_p \frac{DT}{Dt} = -\nabla \cdot \vec{q} + \alpha T Dp/Dt + \bar{\bar{\tau}}' : \nabla \vec{v} \quad (6a)$$

Most of the times energy terms other than the accumulation  $\rho c_p DT/Dt$  and the heat flux  $-\nabla \cdot \vec{q}$  terms, are grouped under a dissipation variable  $\phi$  (energy release per unit volume), as seen in Heat and Mass Transfer.

### Mass transport equation

The (global) mass-transport equation is the continuity equation above; what we deal with here is the species-balance equation in a mixture, ( $dm_{i,CM}/dt = W_i$ , for any species  $i$ , with  $W_i$  being a possible  $i$ -species production term by chemical reactions), for a  $dV$  system. Now, besides substituting  $\phi = \rho_i \vec{v}_i$  in (3), one has to use in (3) the  $i$ -species own velocity,  $\vec{v}_i$ , related to the mass-averaged velocity  $\vec{v}$  by the conservation equation  $\sum \rho_i \vec{v}_i = \rho \vec{v}$ , and the definition of diffusion velocity  $\vec{v}_{di} \equiv \vec{v}_i - \vec{v}$ , what yields:

$$\frac{\partial \rho_i}{\partial t} + \nabla \cdot (\rho_i \vec{v}_i) = w_i = \frac{\partial \rho_i}{\partial t} + \nabla \cdot (\rho_i \vec{v}) + \nabla \cdot (\rho_i \vec{v}_{di}) = \frac{\partial \rho_i}{\partial t} + \nabla \cdot (\rho_i \vec{v}) + \nabla \cdot \vec{j} \quad (7)$$

where  $\vec{j}_i \equiv \rho_i \vec{v}_{di}$  is the flux density of species  $i$  through a (global) fluid particle. With the convective derivative:

$$\frac{D\rho_i}{Dt} + \rho_i \nabla \cdot \vec{v} = w_i - \nabla \cdot \vec{j}_i \quad (7a)$$

$w_i$  being a possible  $i$ -species production term per unit volume.

## Constitutive equations

To solve a Fluid Mechanics problem, i.e. to find the velocity field, pressure field and temperature field  $(\vec{v}, p, T)$  in terms of position and time  $(\vec{x}, t)$ , besides the initial and boundary conditions of the particular problem at hand, the above balance equations of mass, momentum, and energy, must be supplemented with some general constitutive equations that relate the additional variables  $(\rho, \bar{\tau}, e, \vec{q}, \phi, \vec{j}_i, w_i)$  to the main variables  $(\vec{v}, p, T)$ , i.e. the equation of state at equilibrium,  $\rho = \rho(T, p)$  and  $e = e(T, p)$ , and the main one in Fluid Mechanics, a relation between fluid strain-rate and stress, first proposed by Newton in 1687 as,  $\tau = -\mu \partial v / \partial y$ , and in a more general way named Navier-Poisson's law:

$$\bar{\tau} = -p\bar{I} + \bar{\tau}' = -p\bar{I} + 2\mu\bar{\epsilon} + \left(\mu_v - \frac{2}{3}\mu\right)(\nabla \cdot \vec{v})\bar{I} = -p\bar{I} + \mu(\nabla\vec{v} + (\nabla\vec{v})^T) + \left(\mu_v - \frac{2}{3}\mu\right)(\nabla \cdot \vec{v})\bar{I} \quad (8)$$

which enters into the momentum balance as:

$$\nabla \cdot \bar{\tau} = -\nabla p + \nabla \cdot \bar{\tau}' = -\nabla p + \mu(\nabla^2 \vec{v} + \nabla(\nabla \cdot \vec{v})) + \left(\mu_v - \frac{2}{3}\mu\right)\nabla(\nabla \cdot \vec{v}) \stackrel{\nabla \rho = 0}{=} \nabla \cdot \bar{\tau} = -\nabla p + \mu \nabla^2 \vec{v} \quad (8a)$$

Besides state equations and the strain-rate to stress relation, one needs a heat-rate relation (our well-known Fourier's law,  $\vec{q} = -k\nabla T$ , or its extension to multi-component-diffusing systems,  $\vec{q} = -k\nabla T + \sum \rho_i \vec{v}_{di} h_i$ , with the enthalpy of species  $i$ ,  $h_i$ , being deduced from the state equations adopted), plus appropriate dissipation laws for  $\phi$ , plus the mass diffusion rate equations, namely the extended Fick's law and Arrhenius's law:

$$\vec{j}_i = -D_i \nabla (\rho_i + c_s T / \rho) \quad (9)$$

$$w_i = M_i (v_i'' - v_i') \prod \left( \frac{\rho_j}{M_j} \right)^{v_j} B_a \exp\left(\frac{-E_a}{RT}\right) \quad (10)$$

where  $D_i$  is the coefficient of mass diffusion of species  $i$  in a given mixture due to concentration gradients,  $c_s$  is the Soret coefficient of mass diffusion due to thermal gradients (usually negligible),  $w_i$  is the mass of species  $i$  produced (by unit time and volume of mixture) due to chemical reactions,  $v_i''$  and  $v_i'$  the stoichiometric coefficients for the forward and backward reaction considered (one  $w_i$  must be considered for each reactions), and  $B_a$  and  $E_a$  two empirical Arrhenius coefficients. The kinetic theory of gases provides a simple (although sometimes not very accurate) formulation of all the transport

coefficients and equations of state in terms of pressure, temperature and composition, but in practice one usually resorts to tabulated experimental data. As no reaction is to be considered here, and Soret effects are neglected, the only term entering into the mass-diffusion balance is:

$$\nabla \cdot \vec{j}_i = -D_i \nabla^2 \rho_i \quad (11)$$

In summary, substituting these constitutive relations in the balance equations, the partial differential equations that solve a heat and mass transfer problem, with fluid flow but nearly constant density, are:

$$\text{Mass balance (continuity): } \nabla \cdot \vec{v} = 0 \quad (12)$$

$$\text{Species balance: } \frac{Dy_i}{Dt} = D_i \nabla^2 y_i + \frac{w_i}{\rho} \quad (13)$$

$$\text{Momentum balance: } \frac{D\vec{v}}{Dt} = -\frac{\nabla p}{\rho} - g [1 - \alpha(T - T_0)] \vec{i}_z + \nu \nabla^2 \vec{v} \quad (14)$$

$$\text{Energy balance: } \frac{DT}{Dt} = a \nabla^2 T + \frac{\phi}{\rho c_p} \quad (15)$$

where  $y_i \equiv \rho_i / \rho$  is the mass fraction of species  $i$  in the mixture. Notice that the  $i$ -species diffusivity in the mixture,  $D_i$ , the momentum diffusivity (kinematic viscosity)  $\nu$ , and thermal diffusivity,  $a = k / (\rho c_p)$ , all have dimensions of square length divided by time. Finally recall the definition of the convective derivative.  $D() / Dt \equiv \partial() / \partial t + \vec{v} \cdot \nabla()$ , which reduces to  $D() / Dt = \vec{v} \cdot \nabla()$  in the steady-state case.

## Introduction to non-dimensional parameters

In all fields of physical sciences, but particularly in Fluid Mechanics, and above all in Heat and Mass Transfer, there is such a number of parameters interplaying in each problem, that it is most convenient for us to group them if possible, and there is a general principle applicable for that, namely, the Pi-Buckingham Theorem (ASME, 1915), which states that a physical equation with  $N$  variables whose magnitudes can be expressed in terms of  $M$  independent physical units, is equivalent to a non-dimensional physical equation with  $N - M$  non-dimensional variables.

Before attempting that grouping, however, some remarks are appropriate. Firstly, the number of independent physical units,  $M$ , is not a universal invariant but a universal agreement, and the metre-unit (m) could be totally skipped if lengths were measured with the second-unit (s) and the universal law for the speed of light in vacuum  $c = 1$  assumed, instead of giving dimensions to this universal invariant,  $c = 3 \cdot 10^8$  m/s. Secondly, and most important, if working with non-dimensional variables is so advantageous, why most physical subjects are learnt using dimensional magnitudes? The answer is that we, humans, want to compare every magnitude with our own measurements: lengths with the length of our arm span, masses with the mass of a stone we can throw, times with our heart period, and so on, and each of our anthropocentric units we introduce, contributes to one of the  $M$ -basic magnitudes mentioned above (seven in the SI: m, kg, s, K, A, cd, and mol).

For the grouping of dimensional variables to get non-dimensional parameters, one may follow an *ad hoc* approach. For instance, in thermal convection studies, one may reason that the convective coefficient  $h$  must be a function of the fluid properties  $(k, \rho, c_p, \mu)$ , and the characteristic fluid-velocity gradient  $v/L$ , i.e.  $h=h(k, \rho, c_p, \mu, v/L)$ , and say that the combinations  $Nu \equiv hL/k$ ,  $Re \equiv \rho vL/\mu$ , and  $Pr \equiv \mu c_p/k$ , are 'the usual choice', or 'the standard rule', but someone might ask why not another combination, and if there is a rational behind. Adding that the rational is to compare heat convected ( $h\Delta T$ ) against heat conducted if the fluid was quiescent ( $k\Delta T/L$ ), to compare change of momentum ( $(\rho v)v$ ) against viscous stress ( $\mu(v/L)$ ), and to compare momentum diffusivity ( $\nu \equiv \mu/\rho$ ) against thermal diffusivity ( $a=k/(\rho c_p)$ ), may seem enough justification already. But the most conclusive explanation of why those parameters and their meaning, comes from an order-of-magnitude analysis of the general equations presented above, both the balance equations and the boundary conditions, namely:

Thermal boundary condition at a wall. From the definition of heat convection coefficient:

$$\dot{q} \equiv h(T - T_\infty) = -k \nabla_n T \quad \rightarrow \quad h\Delta T \approx k \frac{\Delta T}{L} \quad \rightarrow \quad Nu \equiv \frac{hL}{k} \quad (16)$$

what teaches that a non-dimensional parameter,  $Nu$ , can be defined to measure the ratio of heat flux transferred with convection to that without convection; it is named Nusselt number in honour of the great thermal engineer Wilhelm Nusselt, who introduced it in his 1915 pioneering article "The Basic Laws of Heat"; the 'number' ending is the traditional designation of non-dimensional parameters (no physical units, just the number). In spite of heat convection being always greater than a corresponding heat conduction,  $Nu$  may be smaller than unity if one choose for it a length smaller than the boundary-layer thickness (e.g. when using the diameter for fine wires).

Mass balance. From the continuity equation:

$$\nabla \cdot \vec{v} = 0 \quad \rightarrow \quad \frac{v_x}{L_x} + \frac{v_y}{L_y} = 0 \quad (17)$$

what teaches that, if there is a change of fluid speed along one direction ( $v_x/L_x$ ), it must be a balancing change of fluid speed along another direction (the flow must be at least two-dimensional); i.e., in a one-dimensional flow (in Cartesian coordinates), the speed cannot change along a streamline ( $\partial v/\partial x=0$ ). Notice that the fact that the two velocity-gradients be of equal magnitude does not mean that the longitudinal flow must be of the same order as the transversal flow, the paradigmatic case being the boundary-layer flow to be analysed below, where the longitudinal flow  $v_x$  is dominant, i.e.  $v_y \ll v_x$ , but still  $\partial v_y/\partial y = -\partial v_x/\partial x$ .

Momentum balance. From the longitudinal momentum equation, assuming gravity effects irrelevant, and expanding the convective derivative:



$$\frac{D\vec{v}}{Dt} = -\frac{\nabla p}{\rho} + \nu \nabla^2 \vec{v} \rightarrow \frac{v}{t} + \nu \frac{v}{L} \approx \frac{p}{\rho L} + \nu \frac{v}{L^2} \rightarrow \frac{1}{Sr} \frac{v^2}{L} + \frac{v^2}{L} \approx \frac{p}{\rho L} + \frac{1}{Re} \frac{v^2}{L} \quad \text{with} \quad \begin{cases} Re \equiv \frac{vL}{\nu} \\ Sr \equiv \frac{vt}{L} \end{cases} \quad (18)$$

which can be interpreted in the following way. At least two terms in (18) must be of the same order of magnitude; it is important then to compare each other, and for that purpose several non-dimensional ratios are defined: the Strouhal number to measure the ratio of convective forces per unit volume ( $\rho v^2/L$ ) to inertia forces per unit volume ( $\rho v/t$ ), the Reynolds number to measure the ratio of convective forces per unit volume ( $\rho v^2/L$ ) to viscous forces per unit volume ( $\mu v/L^2$ ), and so on.

Several other non-dimensional parameters are used in heat and mass transfer, as the ratio of momentum diffusivity to thermal diffusivity, named Prandtl number  $Pr = \nu/a$ , the ratio of momentum diffusivity to species diffusivity, named Schmidt number  $Sc = \nu/D_i$ , and so on, all of which will be introduced at due time (Tables 1 and 2 give a compilation), but now we turn to the details of fluid flow.

Table 1. Main non-dimensional parameters in convective heat transfer.

Parameter	Definition	Meaning
Nusselt number	$Nu \equiv \frac{hL}{k}$	Ratio of convective heat flux to conductive heat flux.
Prandtl number	$Pr \equiv \frac{\nu}{a}$	Ratio of momentum diffusivity to thermal diffusivity. Also, thickness ratio between velocity-boundary-layer and thermal-boundary-layer.
Reynolds number	$Re \equiv \frac{vL}{\nu}$	Ratio of flow convection-inertia stress to viscous stress.
Grashof number	$Gr \equiv \frac{\alpha g \Delta T L^3}{\nu^2}$	Ratio of fluid-buoyancy stress to viscous stress.
Rayleigh number	$Ra \equiv \frac{\alpha g \Delta T L^3}{\nu a} = Gr Pr$	Ratio of fluid-buoyancy stress to viscous and thermal stresses.
Peclet number	$Pe = \frac{vL}{a} = Re Pr$	Ratio of flow convection-inertia stress to viscous and thermal stresses.
Stanton number	$St = \frac{h}{\rho v c} = \frac{Nu}{Re Pr}$	Ratio of heat convection to flow thermal capacity.
Strouhal number	$Sr = \frac{\omega L}{v}$	Ratio of flow convection-inertia stress to viscous stress.

Table 2. Main non-dimensional parameters in convective mass transfer.

Parameter	Definition	Meaning
Sherwood number	$Sh = \frac{h_m L}{D_i}$	Ratio of convective mass flux, $\dot{m}_i/A = h_m (\rho_{i,w} - \rho_{i,\infty})$ , to diffusive mass flux, $\dot{m}_i/A = -D_i \nabla \rho_i$ . Notice that if the convection term is written as $\dot{m}_i/A = h_m (y_{i,w} - y_{i,\infty})$ (i.e. including the density in the convective coefficient), then $Sh \equiv h_m L / (\rho D_i)$ .
Smidth number	$Sc = \frac{\nu}{D_i}$	Ratio of momentum diffusivity to solutal diffusivity. Also, thickness ratio between velocity-boundary-layer and solutal-boundary-layer.

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Lewis number	$Le = \frac{a}{D_i} = \frac{Sc}{Pr}$	Ratio of momentum diffusivity to thermal diffusivity. Also, thickness ratio between velocity-boundary-layer and thermal-boundary-layer.
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We focus now on the fluid-mechanics near walls, and will follow on with the analysis of thermal and solutal effects there, but the theory of boundary layers can be applied to other interesting cases like mixing layers, where two parallel streams with different speed, or different temperature, or different composition, meet together. For instance, it can be deduced that, in the laminar regime, similarly to the thickness of the viscous boundary-layer,  $\delta$ , growing parabolically with distance,  $x$ , as  $\delta/x \approx Re_x^{-1/2}$ , the thickness of a thermal boundary-layer (either attached to a wall, or free-flowing between two fluids at different temperature),  $\delta_T$ , grows as  $\delta_T/x \approx (Re_x Pr)^{-1/2}$ , and the thickness of a solutal boundary-layer (either attached to a wall, or free-flowing between two fluids with different composition),  $\delta_S$ , grows as  $\delta_S/x \approx (Re_x Sc)^{-1/2}$ .

## Boundary layer flow

Heat and Mass Transfer by convection, focuses on heat and mass flows at walls; that is why fluid flow near a solid wall (boundary layer flow) is so important.

Here, the general layout of flow fields at high Reynolds-number-flows (those found in most practical problems) should be recalled: the whole fluid flow can be divided in: a) the main nearly-inviscid flow, where viscous effects can be neglected, and b) some thin boundary-layer flows where viscous effects are concentrated; a seminal approach in Fluid Mechanics, first introduced by L. Prandtl in 1904. We focus now on boundary layers attached to walls; free boundary layers, as the mixing layer just mentioned above, or other more complicated shear flows like jets and wakes, present similar behaviour: an initial laminar region that gets unstable at a transition region (where waves appear), with turbulence development further downstream.

## Non-slip condition

The local equilibrium assumption means that, if the observer considers very small systems (e.g. fluid particles, let say  $\approx 10^{-6}$  m in size), with not-too-small time scales (let say  $\approx 10^{-3}$  s), they can be assumed to be at equilibrium, since those times are larger than the relaxation time (which is proportional to size, since its inertia is proportional to its volume, and the forcing is proportional to its surface). Thus, the velocity field cannot have discontinuities, neither within the fluid, not at the boundaries, and thus fluid particles in contact with solid walls must be in mutual equilibrium, i.e. have the same velocity (what implies the non-slip condition, but also the non-detachment condition), the same temperature, and the same chemical potential for each of the species present (not the same concentration, obviously).

## Boundary layer forced-flow over a flat plate

The boundary layer forced-flow over a flat plate is a canonical fluid-mechanics problem where a uniform flow with velocity  $u_\infty$ , meets a flat solid sharp edge aligned with the flow (Fig. 1). In absence of thermal and solutal effects, the presence of the plate at zero incidence only introduces a mechanical perturbation, a shear stress in the direction of flow, due to the non-slip condition, which, at a constant separation from the plate, causes a deceleration of the flow and, as a consequence of the continuity equation, a small

transversal outwards flow that makes the region affected growing. The region affected starts at the entry border and grows along the length of the plate, with longitudinal velocity growing from  $u=0$  to  $u=u_\infty$  across the layer (and small transversal velocities); we may arbitrarily set the thickness of the boundary layer,  $\delta$ , as that where  $u=0.99u_\infty$ , and we want to know its growth rate,  $\delta(x)$ ; let us advance that, after some length, the orderly shear flow (laminar flow) transforms (after some transition region) into a less-ordered turbulent-flow with random velocity-fluctuations, with a thicker boundary layer and a much thinner laminar sub-layer close to the wall (Fig. 1).

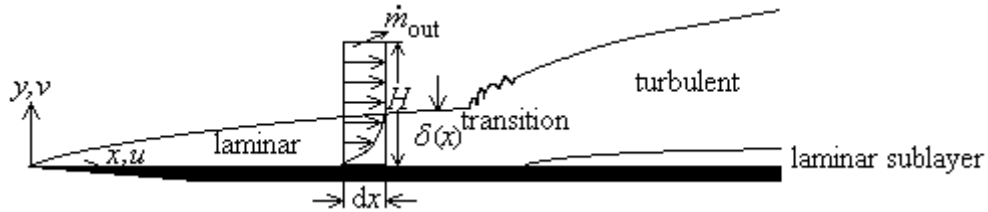


Fig. 1. Structure of the boundary layer flow over a flat plate.

The equations governing the flow over a flat plate, assumed steady, incompressible, and without gravity effects, are the following:

Mass balance (continuity equation (4) with  $\rho=\text{constant}$ ):

$$\nabla \cdot \vec{v} = 0 \rightarrow \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \rightarrow \frac{u_\infty}{L} \approx \frac{v}{\delta} \quad (19)$$

where an order-of-magnitude analysis has also been performed. Assuming the thickness of the boundary layer to be much smaller than the length of the plate under consideration, i.e.  $\delta \ll L$ , the continuity equation shows that transversal velocities are much smaller than longitudinal velocities. Notice that we have assumed the flow to be two-dimensional (really it is quasi-one-dimensional), but, when the flow becomes turbulent, three-dimensional random motions appear.

Momentum balance (equation (5) with  $\rho=\text{constant}$  and  $\partial p/\partial x=0$ ):

$$\frac{D\vec{v}}{Dt} = -\frac{\nabla p}{\rho} + \nu \nabla^2 \vec{v} \rightarrow \begin{cases} u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \nu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \rightarrow \frac{u_\infty^2}{L} \approx \nu \frac{u_\infty}{\delta^2} \rightarrow \frac{\delta}{L} \approx \sqrt{\frac{\nu}{u_\infty L}} \\ u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial y} + \nu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) \rightarrow \frac{\delta}{L} \frac{u_\infty^2}{L} \approx \frac{\partial p}{\partial y} \end{cases} \quad (20)$$

The order-of-magnitude analysis of the transversal momentum-balance shows that transversal pressure variations are negligible (proportional to  $\delta/L$ ), and the longitudinal momentum balance shows that the thickness ratio,  $\delta/L$ , is of order  $(u_\infty L/\nu)^{-1/2}$ , i.e.  $\delta/L \approx Re^{-1/2}$ , which is a most important result, to be compared with the exact solution,  $\delta/L = 4.92 Re^{-1/2}$ , first developed by Blasius (see below). More precisely, the height at which  $u=0.99u_\infty$ , grows parabolically as  $\delta = 4.92(\nu x/u_\infty)^{1/2}$ , e.g., for air with  $\nu=15 \cdot 10^{-6} \text{ m}^2/\text{s}$  moving at  $u_\infty=10 \text{ m/s}$ , the boundary-layer thickness after  $x=1 \text{ m}$  from the leading edge is  $\delta=6 \text{ mm}$ .

Another consequence of (20) can be found applicable to the wall vicinity: since  $u|_{y=0}=0$ , then  $\mu \partial^2 u / \partial y^2|_{y=0} = \partial p / \partial x$  (equal zero for a flat plate), for both the laminar and the turbulent cases!

One of Prandtl's students, P. Blasius, found in 1908 the exact solution by introducing a self-similar variable,  $\eta = y(u_\infty / (\nu x))^{1/2}$ , that transforms the PDE-system into an ordinary differential equation in the auxiliary function (the stream function, such that  $u = \partial \psi / \partial y$  and  $v = -\partial \psi / \partial x$ ),  $\psi(\eta) = (u_\infty \nu)^{1/2} f(\eta)$ , with  $f(\eta) = \int (u/u_\infty) d\eta$ , the equation being:

$$2 \frac{d^3 f}{d\eta^3} + f \frac{d^2 f}{d\eta^2} = 0, \quad \text{with} \quad f|_{\eta=0} = 0, \quad \left. \frac{df}{d\eta} \right|_{\eta=0} = 0, \quad \left. \frac{df}{d\eta} \right|_{\eta=\infty} = 1 \quad (21)$$

which, although not analytically integrable, has a universal solution easily computed numerically, and shown in Fig. 2. The longitudinal speed fraction  $u/u_\infty = \partial f / \partial \eta$  asymptotically grows from 0 at the plate to 1 at infinity, attaining a precise value of 0.99 for  $\eta = 4.92$  (sometimes rounded to  $\eta = 5$ , where  $u/u_\infty = 0.992$ ).

Instead of the exact solution to the boundary-layer equations, an integral approximation may be good enough, i.e., a solution to integral forms of the mass and momentum equations (first developed by von Kármán in 1946), instead of a detailed solution at each point. Let  $u/u_\infty = f(y/\delta)$  be the proposed fitting function (with  $\delta(x)$  the unknown thickness), already verifying the boundary conditions  $f(0) = 0$ ,  $f(1) = 1$ , and  $f'(1) = 0$ ; another condition may be added, because the longitudinal momentum equation (20) at  $y = 0$  is  $0 = \partial^2 u / \partial y^2$ , as said above, and thus  $f''(0) = 0$ , but this is not much important. The integrated equations are obtained for the rectangular control volume of width  $dx$  and height  $H$  sketched in Fig. 1:

$$\left. \begin{aligned} \dot{m} &= \int_0^{\delta(x)} \rho u dy + \rho u_\infty (H - \delta(x)), & \frac{d}{dx} \dot{m} &= \dot{m}_{\text{out}} \\ \dot{p} &= \int_0^{\delta(x)} \rho u^2 dy + \rho u_\infty^2 (H - \delta(x)), & \frac{d}{dx} \dot{p} &= \dot{m}_{\text{out}} u_\infty - \mu \left. \frac{\partial u}{\partial y} \right|_{y=0} \end{aligned} \right\} \left. \begin{aligned} \frac{d}{dx} \int_0^{\delta(x)} \rho u^2 dy &= \frac{d}{dx} \int_0^{\delta(x)} \rho u u_\infty dy - \mu \left. \frac{\partial u}{\partial y} \right|_{y=0} \end{aligned} \right\} \quad (22)$$

which can be more explicitly formulated in terms of  $f = u/u_\infty$  and  $\eta = y/\delta(x)$  as:

$$\int_0^{\delta(x)} \frac{d}{dx} ((1-f)f) dy = \frac{\nu}{u_\infty^2} \left. \frac{df}{dy} \right|_{y=0} \Rightarrow \left[ \int_0^1 (2f-1) \frac{df}{d\eta} \eta d\eta \right] \frac{d\delta}{dx} = \frac{\nu}{u_\infty \delta} \left. \frac{df}{d\eta} \right|_{\eta=0} \quad (23)$$

Now, an explicit  $f(\eta)$  will yield an explicit  $\delta(x)$  from (23). For instance, if we try the simplest polynomial verifying the three boundary conditions above,  $f = 2\eta - \eta^2$  (i.e.  $u/u_\infty = 2(y/\delta) - (y/\delta)^2$ ), we get the differential equation  $(2/15)\delta'/dx = 2\nu/(u_\infty \delta)$ , which, with the condition  $\delta(0) = 0$  (the layer starts at the leading edge), finally yields the result sought:  $\delta(x) = (30\nu x/u_\infty)^{1/2}$ , i.e. the boundary-layer-thickness grows parabolically with the distance to the edge. We can now find the un-stretched longitudinal velocity field,  $u(x,y) = u_\infty (2y/\delta(x) - (y/\delta(x))^2)$ , the longitudinal velocity slope at the wall,  $\partial u / \partial y|_{y=0} = (u_\infty / \delta) \partial f / \partial \eta|_{\eta=0} = 2(u_\infty / \delta) = 2u_\infty^2 / (30\nu x u_\infty)^{1/2}$ , the wall shear stress  $\tau = \mu \partial u / \partial y|_{y=0}$ , and the drag

coefficient,  $c_f$ , is defined by  $\tau=c_f\rho u_\infty^2/2$ , which is  $c_f=(4/30^{1/2})/(u_\infty x/\nu)^{1/2}$ . The transversal velocity profile can be found from continuity equation (19) as  $v(x,y)=\int -(\partial u/\partial x)dy=(5/6)^{1/2}u_\infty(u_\infty x/\nu)^{-1/2}(3\eta^2-2\eta^3)$ .

If we add the fourth boundary condition stated above,  $\partial^2 u/\partial y^2|_{y=0}=0$ , we need a cubic polynomial, which is  $f=(3\eta-\eta^3)/2$ , and new explicit values can be found as just shown.. Table 3 gives a summary of those results, and Fig. 2 the corresponding velocity profiles. Notice, by the way, that the transversal velocity  $v(y)$  at a stage  $x$  grows in a S-shape from 0 (with  $dv/dy=0$ ) to a maximum  $v(\delta)=0.86u_\infty/(Re_x)^{1/2}$  (0.86 for Blasius solution;  $0.91=(5/6)^{1/2}$  with the simple fitting above), and remains with that value outside the boundary layer, contrary to some intuitive reasoning telling that it should vanished outside the boundary layer in the undisturbed flow; the explanation is that, with the incompressible fluid model, there is no undisturbed flow, the perturbations travelling in all directions instantaneously, and the  $\dot{m}_{out}$  contribution always exists (in reality, the whole boundary-layer model relies on the  $\delta \ll x$  assumption, not valid near the leading edge and far outside the boundary layer).

Table 3. Comparison of different solutions to the laminar boundary layer flow over a flat plate.

Code (*)	Solution, $f$ $f \equiv u/u_\infty$	Thickness coeff., $a$ , in $\delta/x \equiv a/Re_x^{1/2}$	Slope coeff., $b$ , in $\partial u/\partial y \equiv b Re_x^{1/2} u_\infty/x$	Friction coeff.,** $c$ , in $c_{f,x}=c/Re_x^{1/2}$	Coeff., $d$ , in $v_\infty/u_\infty=d/Re_x^{1/2}$
4	$2(y/\delta)-(y/\delta)^2$	$\sqrt{30}$ =5.48	$\sqrt{2/15}$ =0.365	$4/\sqrt{30}$ =0.730	$\sqrt{5/6}$ =0.913
1	$(3(y/\delta)-(y/\delta)^3)/2$	$\sqrt{3640/13}$ =4.64	$\sqrt{117/1120}$ =0.323	$\sqrt{117/280}$ =0.646	$\sqrt{315/416}$ =0.870
2	$\sin(\pi(y/\delta)/2)$	$\pi\sqrt{2/(4-\pi)}$ =4.80	$\sqrt{(4-\pi)/8}$ =0.328	$\sqrt{(4-\pi)/2}$ =0.655	$(2-\pi)/\sqrt{8-2\pi}$ =0.871
3	Exact sol.. (21)	4.92***	0.332	0.664	0.86

\*Codes for Fig. 2. \*\*Defined from  $\tau=\mu\partial u/\partial y|_{y=0}=c_f\rho u_\infty^2/2$ . \*\*\*Exact solution when  $u/u_\infty=0.99$  (it extends from  $y=0$  to  $\infty$ , whereas the others extend from  $y=0$  to  $\delta$ ); in many books, this 4.92 is rounded to 5.0.

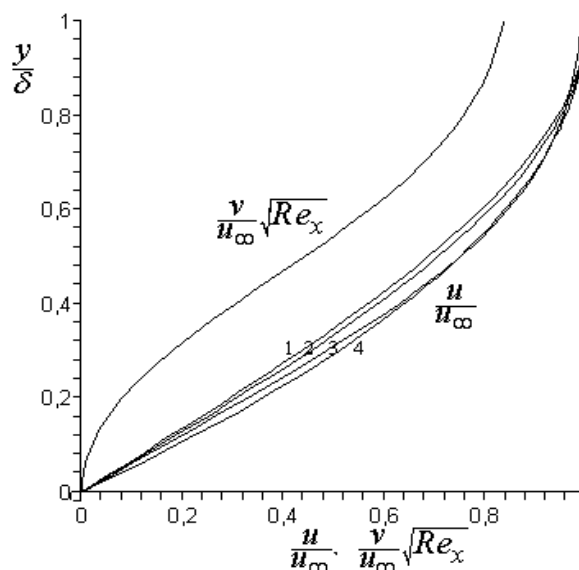


Fig. 2. Non-dimensional velocity profiles inside the boundary layer. Four models are shown for the longitudinal velocity profile,  $u(x,y)$  (see details in Table 3), and only the exact profile for the transversal velocity profile,  $v(x,y)$ , with  $Re_x \equiv u_\infty x/\nu$ .

Notice that the choice of reference frame modifies the expression of the velocity profile, and, for instance, the profile  $u/u_\infty=2(y/\delta)-(y/\delta)^2$  (number 4 in Fig. 2) refers to the origin at the plate, whereas if the origin is set at the free-end of the boundary layer, the same profile would read  $u/u_\infty=1-(y/\delta)^2$ . By the way, the latter origin is more convenient for fully-developed flow in pipes and two-dimensional ducts, where the boundary layers meet at the centre and, with the origin there, the expression  $u/u_0=1-(y/\delta)^2$  is valid for the whole duct; notice the change from  $u_\infty$  to  $u_0$ , the speed at the centre line, which is (3/2)-times the average speed in two-dimensional ducts, and twice the average speed in circular pipes. This parabolic velocity profile (named Poiseuille flow) becomes more uniform in turbulent flow, where it can be approximated by a higher-power law  $u/u_\infty=1-(y/\delta)^n$  with  $n$  between 6 and 10 ( $n=7$  is the most common).

Besides the normal boundary-layer thickness defined with  $u(y)=0.99u_\infty$ , two other related variables are sometimes used to quantify boundary-layer thickness: the displacement thickness  $\delta^*(x)$  defined by  $\delta^* \equiv (1/u_\infty) \int (u_\infty - u(y)) dy$ , and the momentum thickness  $\theta(x)$  defined by  $\theta \equiv (1/u_\infty^2) \int u(y)(u_\infty - u(y)) dy$ . For a laminar boundary layer over a flat plate with no pressure gradient,  $\delta^* \approx \delta/3$  and  $\theta \approx 2\delta/15$ .

All the models developed above, only apply to laminar boundary layer flow over a flat plate. In practice, there is an initial length with laminar flow for both sharp and rounded blunt leading edges (i.e. provided there is not flow separation at the edge), followed on by a transition region starting at some  $x$  such that  $Re_x=(0.3..1) \cdot 10^6$  (with very smooth plates, laminar flows up to  $Re_x=3 \cdot 10^6$  have been achieved), and finally ending in a turbulent flow downstream. For most engineering problems it is assumed that the transition region is abrupt, and that the laminar region spans from  $x=0$  to  $x=0.5 \cdot 10^6 \cdot \nu/u_\infty$  (corresponding to a standard critical value of  $Re=0.5 \cdot 10^6$ ), and the turbulent one starts there and extends beyond (it actually depends on plate roughness and turbulence level of the entry-flow. Turbulent thickness cannot be analytically modelled (cross-coupling velocity terms appear in the momentum equation, Newton's law of friction  $\tau = \mu \partial u / \partial y|_{y=0}$  is no longer valid, and so on), and empirical correlations, based on the momentum-energy Reynolds analogy (explained below), are used; traditional correlations are presented in Table 4, in comparison with their laminar counterparts.

Table 4. Comparison of laminar boundary-layer characteristics model with turbulent ones (see Table 3).

	Velocity profile	Layer thickness	Friction coefficient*
Laminar** $Re_x < 0.5 \cdot 10^6$	$\frac{u}{u_\infty} = 1 - \left(\frac{y}{\delta}\right)^2$	$\frac{\delta}{x} = \frac{4.92}{Re_x^{1/2}}$	$c_{f,x} = \frac{0.66}{Re_x^{1/2}}, c_{f,L} = \frac{1.33}{Re_L^{1/2}}$
Turbulent*** $0.5 \cdot 10^6 < Re_x < 10 \cdot 10^6$	$\frac{u}{u_\infty} = 1 - \left(\frac{y}{\delta}\right)^7$	$\frac{\delta}{x} = \frac{0.38}{Re_x^{1/5}}$	$c_{f,x} = \frac{0.059}{Re_x^{1/5}}, c_{f,L} = \frac{0.074}{Re_L^{1/5}}$

\*Local friction coefficient is defined by  $\tau(x) = c_{f,x} \rho u_\infty^2 / 2$ , whereas global friction coefficient is defined by  $(1/L) \int \tau(x) dx = c_{f,L} \rho u_\infty^2 / 2$ . \*\*Blasius laminar velocity profile code 4 in Table 3 and Fig. 2, but with exact coefficients for  $\delta$  and  $c_f$ . Notice the change of  $y$ -coordinate origin and sense from Table 3 and Fig. 2. \*\*\*Prandtl turbulent model as calibrated by Schlichting;  $x$ -coordinate origin at transition point.

Notice that turbulent thickness (Table 4) is not defined so precisely as laminar thickness, where the separation at which  $u(y)=0.99u_\infty$ , is neat; in turbulent boundary layers, large eddies are created and burst,

causing typical protuberances up to  $1.2\delta$  and depressions down to  $0.5\delta$ . In any case, it can be concluded that turbulent thickness is always larger than laminar thickness and grows quicker. Laminar-to-turbulent transition (LTT) depends a lot in the pressure gradient in non-flat surfaces (to be studied [aside](#)); even more, there can be a turbulent-to-laminar transition in the strongly favourable pressure gradient that occurs in a converging nozzle (relaminarization).

A general warning on using empirical correlations is to be careful about the application range: all empirical correlations are limited in scope, and the most accurate, the narrower their applicability range.

### Thermal boundary layer and solutal boundary layer in a forced-flow over a flat plate

Analogous to the velocity boundary layer due to the jump from the non-slip condition to the free-stream flow, a thermal-boundary-layer appears if there is a difference from wall-temperature to free-flow-temperature, and a solutal boundary layer appears if there is a difference from wall-concentration of a solute to its free-flow concentration.

The governing balance equations for the general case of flow-, thermal-, and solutal-boundary layers are:

$$\nabla \cdot \vec{v} = 0 \rightarrow \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (24)$$

$$\frac{D\vec{v}}{Dt} = -\frac{\nabla p}{\rho} + \nu \nabla^2 \vec{v} \rightarrow u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2} \quad (25)$$

$$\frac{DT}{Dt} = a \nabla^2 T + \frac{\phi}{\rho c_p} \rightarrow u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = a \frac{\partial^2 T}{\partial y^2} \quad (26)$$

$$\frac{Dy_i}{Dt} = D_i \nabla^2 y_i + \frac{w_i}{\rho} \rightarrow u \frac{\partial y_i}{\partial x} + v \frac{\partial y_i}{\partial y} = D_i \frac{\partial^2 y_i}{\partial y^2} \quad (27)$$

Boundary conditions can be layout in a similar way to the velocity boundary layer above-explained, what shows that in the case of  $Pr \equiv \nu/a=1$ , the function  $(T-T_w)/(T_\infty-T_w)$  has the same shape as the already-known  $u/u_\infty$  profile, and, in the case of  $Sc \equiv \nu/D_i=1$ , the function  $(y_i-y_{iw})/(y_{i\infty}-y_{iw})$  has the same shape as the already-known  $u/u_\infty$  profile; see Table 3 for several approximations.

The main goal in heat convection is founding  $h$  (or  $Nu$ , in non-dimensional variables), which with the above thermal-boundary-layer model yields:

$$h \equiv \frac{\dot{q}}{T_w - T_\infty} = \frac{-k \partial T / \partial y|_{y=0}}{T_w - T_\infty} \rightarrow Nu_x \equiv \frac{hx}{k} = \frac{x \partial T / \partial y|_{y=0}}{T_\infty - T_w} \stackrel{\frac{T-T_w}{T_\infty-T_w}=f\left(\frac{y}{\delta}\right)}{=} \left. \frac{x}{\delta} \frac{\partial f(\eta)}{\partial \eta} \right|_{\eta=0} = 0.33 \sqrt{Re_x} \quad (28)$$

where the exact Blasius solution is used (see Table 3); using instead the simplest model  $(T-T_w)/(T_\infty-T_w) = u/u_\infty = 2(y/\delta) - (y/\delta)^2 = 1 - (1-y/\delta)^2$ , one gets  $Nu_x = 0.26 \sqrt{Re_x}$ . Instead of the local Nusselt number, the global-average value over a whole plate of length  $L$ ,  $Nu_L \equiv (1/L) \int Nu_x dx = 2Nu_{x=L}$  can be used.

As above, the turbulent case cannot be analytically solved, but the equivalence between thermal boundary-layer and velocity boundary-layer (for  $Pr=1$ ), allows to compute the temperature gradient at the wall in terms of the velocity gradient at the wall, much easier to measure, what gives  $Nu_x=c_f Re_x/2$ , called Reynolds analogy, although a modified Reynolds analogy, named Reynolds-Colburn or Colburn-Chilton analogy, is commonly used (see below).

An entirely similar with the convection of species  $i$  in a mixture, where the mass-convection coefficient,  $h_m$ , and Sherwood number,  $Sh$ , are defined in terms of the mass-flow-rate of species  $i$  at the interface,  $\vec{j}_i$ , as:

$$h_m \equiv \frac{\vec{j}_i}{\rho_{iw} - \rho_{i\infty}} = \frac{-D_i \partial \rho_i / \partial y|_{y=0}}{\rho_{i\infty} - \rho_{iw}} \rightarrow Sh_x \equiv \frac{h_m x}{D_i} = \frac{x \partial \rho_i / \partial y|_{y=0}}{\rho_{i\infty} - \rho_{iw}} \stackrel{\frac{\rho_i - \rho_{iw}}{\rho_{i\infty} - \rho_{iw}} = f\left(\frac{y}{\delta}\right)}{=} \left. \frac{x}{\delta} \frac{\partial f(\eta)}{\partial \eta} \right|_{\eta=0} = 0.33 \sqrt{Re_x} \quad (29)$$

The problem now is to find the solution for the thermal boundary layer in the laminar case but for  $Pr \neq 1$ , and for the solutal boundary layer when  $Sc \neq 1$ . Besides, we may want to consider thermal or solutal convection to start somewhere downstream, at  $x=x_0$ , and not precisely at the leading edge of the plate,  $x=0$ .

For  $Pr > 1$ , thermal diffusivity (i.e. penetration) is smaller than momentum diffusivity ( $a < \nu$ ), and consequently the thermal boundary layer,  $\delta_T$ , is thinner than the velocity boundary layer,  $\delta$ . Let us measure the ratio by  $\zeta \equiv \delta_T / \delta$ . The integral method, applied before to the velocity boundary layer, for the rectangular control volume of width  $dx$  and height  $H$  sketched in Fig. 1, gives now:

$$(30) \quad \left. \begin{aligned} \dot{m} &= \int_0^{\delta(x)} \rho u dy + \rho u_\infty (H - \delta(x)), & \frac{d}{dx} \dot{m} &= \dot{m}_{out} \\ \dot{p} &= \int_0^{\delta(x)} \rho u^2 dy + \rho u_\infty^2 (H - \delta(x)), & \frac{d}{dx} \dot{p} &= \dot{m}_{out} u_\infty - \mu \left. \frac{\partial u}{\partial y} \right|_{y=0} \\ \dot{e} &= \int_0^{\delta(x)} \rho u e dy + \rho u_\infty e_\infty (H - \delta(x)), & \frac{d}{dx} \dot{e} &= \dot{m}_{out} e_\infty - k \left. \frac{\partial T}{\partial y} \right|_{y=0} \end{aligned} \right\} \left. \begin{aligned} \frac{d}{dx} \int_0^{\delta(x)} \rho u^2 dy &= \frac{d}{dx} \int_0^{\delta(x)} \rho u u_\infty dy - \mu \left. \frac{\partial u}{\partial y} \right|_{y=0} \\ \frac{d}{dx} \int_0^{\delta(x)} \rho u e dy &= \frac{d}{dx} \int_0^{\delta(x)} \rho u e_\infty dy - k \left. \frac{\partial T}{\partial y} \right|_{y=0} \end{aligned} \right\}$$

Using the simplest approximation  $u/u_\infty = f(\eta) = 2(y/\delta) - (y/\delta)^2$  (see Table 3), putting energy proportional to temperature,  $e = cT$ , a corresponding profile  $(T - T_w)/(T_\infty - T_w) = 2(y/\delta_T) - (y/\delta_T)^2 = f(\eta/\zeta)$ , and  $\zeta \equiv \delta_T / \delta$ , one gets:

$$\left\{ \begin{aligned} \frac{d}{dx} \int_0^{\delta(x)} (u(u - u_\infty)) dy &= -\nu \left. \frac{\partial u}{\partial y} \right|_{y=0} \rightarrow \frac{d}{dx} \int_0^{\delta(x)} f(\eta)(f(\eta) - 1) dy = -\frac{\nu}{u_\infty^2} \left. \frac{\partial u}{\partial y} \right|_{y=0} \\ \frac{d}{dx} \int_0^{\delta_T(x)} (u(T - T_\infty)) dy &= -a \left. \frac{\partial T}{\partial y} \right|_{y=0} \rightarrow \frac{d}{dx} \int_0^{\delta_T(x)} f(\eta) \left( f\left(\frac{\eta}{\zeta}\right) - 1 \right) dy = -\frac{a}{u_\infty (T_\infty - T_w)} \left. \frac{\partial T}{\partial y} \right|_{y=0} \end{aligned} \right. \quad (31)$$



where the energy integral is limited to  $\delta_T$  because  $T-T_\infty$  is zero outside. Performing the substitutions and integration, (31) yields:

$$\left\{ \begin{array}{l} \frac{d}{dx} \left( \frac{-2\delta(x)}{15} \right) = -\frac{v}{u_\infty} \frac{2}{\delta(x)} \rightarrow \delta(x) = \sqrt{30} \sqrt{\frac{vx}{u_\infty}} \\ \frac{d}{dx} \left( \frac{5\zeta(x)^2 \delta(x)}{30} - \frac{\zeta(x)^3 \delta(x)}{30} \right) = -\frac{a}{u_\infty} \frac{2}{\zeta(x)\delta(x)} \end{array} \right\} \left[ (20-6\zeta)x \frac{d\zeta}{dx} + 5\zeta - \zeta^2 \right] \zeta^2 = \frac{4}{Pr} \quad (32)$$

i.e., the momentum integral gives the thickness law for the velocity boundary layer, which is substituted in the energy integral to get, either a constant  $\delta_T/\delta$ -relation if only the leading terms are kept, namely  $5\zeta^2=4/Pr$ , or, if an initial condition  $\zeta(x_0)=0$  is imposed:

$$\zeta(x) \equiv \frac{\delta_T}{\delta} = 0.93 Pr^{-1/3} \left[ 1 - \left( \frac{x_0}{x} \right)^{3/4} \right]^{-1/3} \stackrel{x_0 \rightarrow 0}{=} 0.93 Pr^{-1/3} \quad (33)$$

If, instead of model 4 (see Table 3), the more refined model 1 is used, the only changes are the change in the coefficients:  $\delta=4.64(vx/u_\infty)^{1/2}$  instead of  $\delta=5.48(vx/u_\infty)^{1/2}$ , and 1.02 instead of 0.93 in (33). Besides the thermal thickness, the slope of the thermal profile at the wall is important; with model 4, for which we found  $du/dy|_{y=0}=0.365(vx/u_\infty)^{1/2}$ , now we get, for  $x_0=0$ ,  $dT/dy|_{y=0}=0.39(T_\infty-T_w)Re_x^{1/2}Pr^{1/3}/x$ , which is not too far from the most precise coefficient found by Pohlhausen in 1921 (0.33 instead of  $(270)^{-1/6}=0.39$ ). In the traditional non-dimensional form, retaining the possibility of the temperature jump starting somewhere downstream, at  $x=x_0$ , one has:

$$Nu_x \equiv \frac{hx}{k} = \frac{x \partial T / \partial y|_{y=0}}{T_\infty - T_w} = 0.33 Pr^{1/3} Re_x^{1/2} \left[ 1 - \left( \frac{x_0}{x} \right)^{3/4} \right]^{-1/3} \quad (34)$$

Again, the global-average value of the Nusselt number over a whole plate of length  $L$ , is often used; for  $x_0=0$ ,  $Nu_L \equiv (1/L) \int Nu_x dx = 2Nu_{x=L}$ .

Notice that the above result comes from a global energy balance in the whole of the thermal layer, thus, an integral average of the fluid properties must be used, and not just their values at wall conditions; these 'film averaged' values are usually computed just as the algebraic mean of the values at wall conditions and at bulk conditions (here the undisturbed conditions), i.e.  $T_{\text{film}} \equiv (T_w + T_\infty)/2$ .

Pohlhausen correlation (34), although deduced for  $Pr > 1$ , has been found to be accurate for  $0.6 < Pr < 60$ , but not enough for the very high Prandtl numbers exhibited by some oils and silicones, and for  $Pr \ll 1$  typical of liquid metals. An extension to Pohlhausen correlation in the whole range of Prandtl numbers was made by Churchill and Ozoe in 1973 in the form:

$$Nu_x = \frac{0.34 Pr^{1/3} Re_x^{1/2}}{\left[1 + \left(\frac{0.047}{Pr}\right)^{2/3}\right]^{1/4}} \quad (35)$$

valid also for the case when the heat flux density at the wall is kept constant (instead of the wall temperature), if the coefficient 0.34 is changed to 0.46, and the coefficient 0.047 is changed to 0.021.

From the local Nusselt number  $Nu_x$ , the local convective coefficient is deduced,  $h_x = kNu_x/x$ . For practical work it may be good enough to use a global convective coefficient  $h_L$  to be deduced from a global Nusselt number  $Nu_L$ . Notice, however, that  $h_L$  is the average value of  $h_x$ , but  $Nu_L$  is not the average of  $Nu_x$ ; e.g., from (34) with  $x_0=0$ :

$$\begin{aligned} Nu_L &\neq \frac{1}{L} \int_0^L Nu_x dx = \frac{1}{L} \int_0^L 0.33 Pr^{1/3} \left(\frac{u_\infty x}{\nu}\right)^{1/2} dx = \frac{2}{3} Nu_x|_{x=L} \\ h_L &= \frac{1}{L} \int_0^L h_x dx = \frac{1}{L} \int_0^L \frac{kNu_x}{x} dx = \frac{1}{L} \int_0^L \frac{k}{x} 0.33 Pr^{1/3} \left(\frac{u_\infty x}{\nu}\right)^{1/2} dx = 2h_x|_{x=L} \end{aligned} \quad (36)$$

## Reynolds analogy between momentum and energy equations

Reynolds analogy is based on the similarity between momentum, heat, and mass transfer from the general balance equations:

$$\frac{D\vec{v}}{Dt} \stackrel{\nabla p=0}{=} \nu \nabla^2 \vec{v}, \quad \frac{DT}{Dt} \stackrel{\phi=0}{=} a \nabla^2 T, \quad \frac{Dy_i}{Dt} \stackrel{w_i=0}{=} D_i \nabla^2 y_i \quad \rightarrow \quad \frac{1}{\nu} \frac{\nabla v}{v} = \frac{1}{a} \frac{\nabla T}{\Delta T} = \frac{1}{D_i} \frac{\nabla y_i}{\Delta y_i} \quad (37)$$

It follows from (37) that, if  $\nu=a=D_i$ , then the scaled functions would be identical,  $u(x,y)/\Delta u = T(x,y)/\Delta T = y_i(x,y)/\Delta y_i$  (with  $\Delta u \equiv u_\infty - 0$ ,  $\Delta T \equiv T_\infty - T_0$ , and  $\Delta y_i \equiv y_{i\infty} - y_{i0}$ ), their slopes at the wall identical too, and thus, with the definition of the Fanning factor,  $c_f$ , and the convective coefficients, for laminar flows, one gets:

$$\left. \begin{aligned} Nu &\equiv \frac{hL}{k}, \quad h = \frac{k \nabla T|_w}{\Delta T} \Rightarrow \frac{\nabla T|_w}{\Delta T} = \frac{Nu}{L} \\ c_f &\equiv \frac{\tau}{\frac{1}{2} \rho u_\infty^2} = \frac{\mu \nabla u|_w}{\frac{1}{2} \rho u_\infty^2} \Rightarrow \frac{\nabla u|_w}{\Delta u} = \frac{u_\infty c_f}{2\nu} \end{aligned} \right\} \Rightarrow \frac{\frac{\nabla T|_w}{\Delta T}}{\frac{\nabla u|_w}{\Delta u}} = 1 = \frac{\frac{Nu}{L}}{\frac{u_\infty c_f}{2\nu}} = \frac{Nu}{\frac{c_f}{2} Re} \quad (38)$$

i.e.  $Nu = (c_f/2)Re$ , known as Reynolds analogy (he deduced it in 1874), and valid for laminar flows (to apply Newton's law of friction) and  $Pr=Sc=1$  (for  $\nu=a=D_i$ ).

The influence of  $Pr \neq 1$  (and/or  $Sc \neq 1$ ) can be retained by stretching the transversal dimension differently for each function in (36) to absorb the respective coefficient; i.e., now the functions which are identical are,  $u(x,y/\nu)/\Delta u = T(x,y/a)/\Delta T = y_i(x,y/D_i)/\Delta y_i$ , and the slopes which are identical are.

$\nabla u(x,y/v)/\Delta u = v \nabla u(x,y)/\Delta u = \nabla T(x,y/a)/\Delta T = a \nabla T(x,y)/\Delta T = \nabla y_i(x,y/D_i)/\Delta y_i = D_i \nabla y_i(x,i)/\Delta y_i$ , Thence, (38) becomes:

$$\left. \begin{aligned} Nu &\equiv \frac{hL}{k}, \quad h = \frac{k \nabla T|_w}{\Delta T} \Rightarrow a \frac{\nabla T|_w}{\Delta T} = a \frac{Nu}{L} \\ c_f &\equiv \frac{\tau}{\frac{1}{2} \rho u_\infty^2} = \frac{\mu \nabla u|_w}{\frac{1}{2} \rho u_\infty^2} \Rightarrow v \frac{\nabla u|_w}{\Delta u} = v \frac{u_\infty c_f}{2v} \end{aligned} \right\} \frac{a \frac{\nabla T|_w}{\Delta T}}{v \frac{\nabla u|_w}{\Delta u}} = 1 = \frac{a \frac{Nu}{L}}{v \frac{u_\infty c_f}{2v}} = \frac{Nu}{\frac{c_f}{2} Re Pr} \quad (39)$$

i.e.  $Nu = (c_f/2) Re Pr$ . However, if we compare the exact solutions to the laminar boundary layer obtained above (Eq. (34) and Table 4), we obtained a more accurate  $Pr$ -correction to Reynolds analogy:

$$\left. \begin{aligned} Nu_x^{x_0=0} &= 0.33 Pr^{1/3} Re_x^{1/2} \\ c_{f,x} &= \frac{0.66}{Re_x^{1/2}} \end{aligned} \right\} \Rightarrow Nu_x = \frac{c_{f,x}}{2} Re_x Pr^{1/3} \Rightarrow St \equiv \frac{Nu_x}{Pr Re_x} = \frac{c_{f,x}}{2 Pr^{2/3}} \quad (40)$$

which is known as Chilton-Colburn analogy (1934);  $St$  is a combined parameter named Stanton number. Although (40) has been developed only for a laminar-boundary-layer flow over a flat plate, it applies with good accuracy to both laminar and turbulent flows over flat plates in the  $0.6 < Pr < 60$ , and even to any turbulent flow with pressure gradients, but not to laminar flows with  $\nabla p \neq 0$  (i.e. Colburn-Chilton analogy can be applied to any turbulent flow, but only to laminar flows over flat plates, not to laminar flows in pipes or around bodies). A compilation of heat-transfer correlations in forced convection over a flat plate is presented in Table 5. A note on correlations for turbulent flow is required: the local Nusselt number  $Nu_x$  is used to get the local convective coefficient using  $h_x = k Nu_x / x$  with  $x$  measured from the leading edge of the plate (not from the start of the turbulent layer), whereas the global Nusselt number  $Nu_L$  is used to get the global convective coefficient using  $h_L = k Nu_L / L$  which includes both the laminar zone and the turbulent zone:

$$\begin{aligned} h_L &= \frac{1}{L} \int_0^L h_x dx = \frac{1}{L} \left( \int_0^{x_{tr}} h_{x,lam} dx + \int_{x_{tr}}^L h_{x,turb} dx \right) = \frac{1}{L} \left( \int_0^{x_{tr}} \frac{k}{x} 0.33 Pr^{1/3} \left( \frac{u_\infty x}{v} \right)^{1/2} dx + \int_{x_{tr}}^L \frac{k}{x} 0.03 Pr^{1/3} \left( \frac{u_\infty x}{v} \right)^{4/5} dx \right) = \\ &= \frac{k Pr^{1/3}}{L} \left( 2 \cdot 0.33 Re_{tr}^{1/2} + \frac{5}{4} 0.030 (Re_L^{4/5} - Re_{tr}^{4/5}) \right) \xrightarrow{Re_{tr}=0.5 \cdot 10^6} h_L = \frac{k Pr^{1/3}}{L} (467 + 0.037 (Re_L^{4/5} - 36200)) = \\ h_L &= \frac{k Pr^{1/3}}{L} (0.037 Re_L^{4/5} - 870) \end{aligned} \quad (41)$$

Table 5. Heat transfer correlations in forced convection over a flat plate.

Flow regime	Correlation
Laminar $Re_x < Re_{tr} = 0.5 \cdot 10^6$ (i.e. $x < x_{tr} = 0.5 \cdot 10^6 v / u_\infty$ )	If $0.6 < Pr < 500$ (modified Pohlhausen equation): $Nu_x = 0.33 Pr^{1/3} Re_x^{1/2} \left[ 1 - \left( \frac{x_0}{x} \right)^{3/4} \right]^{-1/3}$
(coefficients shown are for constant $T_{wall}$ ; for constant $q_{wall}$ , change	

coeff. 0.33 to 0.45, coeff. 0.66 to 1.32)	$Nu_L = 0.66 Pr^{1/3} Re_L^{1/2} \frac{1 - \left(\frac{x_0}{L}\right)^{3/4}}{1 - \left(\frac{x_0}{L}\right)}$
	If $Pr < 0.05$ (liquid metals): $Nu_x = 0.56 (Pr Re_x)^{1/2}$ $Nu_L = 1.12 (Pr Re_L)^{1/2}$
Turbulent $0.2 \cdot 10^6 < Re < 10^8$ (either constant $T_{wall}$ or constant $q_{wall}$ ). The global value includes the laminar contribution at $Re_x < 0.5 \cdot 10^6$ .	If $0.6 < Pr < 60$ : $Nu_x = 0.030 Pr^{1/3} Re_x^{4/5}$ $Nu_L = (0.037 Re_L^{4/5} - 870) Pr^{1/3}$

An entirely similar analogy can be applied to mass convection:

$$\frac{Sh_x}{Sc Re_x} = \frac{c_{f,x}}{2Sc^{2/3}} \quad (42)$$

valid for  $0.6 < Sc < 3000$  and both laminar and turbulent flows.

Notice, by the way, that other types of friction factors different to the Fanning friction factor,  $c_f$ , defined in (37) are often used in some cases, particularly in [pipe flow](#), namely the Darcy friction factor,  $\lambda = 4c_f$  (named  $f$  sometimes).

### Steps to solve heat and mass convection problems

To solve a heat or mass convection problem, the following steps are usually followed:

- Make a quick order-of-magnitude analysis using typical values as from Table 6.
- Geometry characterisation, i.e. try to reduce the geometry to a canonical case (e.g. flat wall, cylinder, tube bank...).
- Reference conditions characterisation, i.e. identify type of fluid, and estimate fluid properties at estimated film-averaged conditions.
- Reynolds number computation, to discern the flow type.
- Selection of the appropriate non-dimensional correlation (local or global).

Table 6. Order of magnitude of convection coefficient,  $h$ , for typical configurations.

Configuration	Typical value of $h$ [W/(m <sup>2</sup> ·K)]	Typical range, $h$ [W/(m <sup>2</sup> ·K)]
Natural convection in air (<1 m/s)	10	2..20
Forced convection in air (>5 m/s)	50	20..200
Natural convection in water (<0.1 m/s)	200	10..1000
Forced convection in water (>0.5 m/s)	5000	50..20 000
Natural convection boiling in water	4000	1000..10 000
Forced convection boiling in water	30 000	10 000..50 000
Natural convection condensation in water	6000	2000..10 000
Forced convection condensation in water	50 000	10 000..100 000

Exercise 1. Find the heat transfer rate from a flat plate of  $2 \times 0.5 \text{ m}^2$  at  $100 \text{ }^\circ\text{C}$  when ambient air is blowing over the plate at  $10 \text{ m/s}$ .

Solution. It is a boundary-layer convection problem.

- Order-of-magnitude analysis:  $\dot{Q} = hA\Delta T$ , with  $h = 50 \text{ W}/(\text{m}^2 \cdot \text{K})$  (forced convection in air),  $A = 2 \times 0.5 = 1 \text{ m}^2$ , and  $\Delta T = 100 - 20 = 80 \text{ K}$ , what yields  $\dot{Q} = hA\Delta T = 50 \cdot 1 \cdot 80 = 4 \text{ kW}$ , but this may be double or half, so we follow on with a finer analysis.
- Geometry. The direction of flow in the plane is not specified; we will try both extremes: flow along the shortest side, and flow along the longest side (we could work out any direction by considering longitudinal strips and integrating).
- Reference conditions. Assuming ambient air at  $15 \text{ }^\circ\text{C}$  and  $100 \text{ kPa}$ , film-averaged temperature is  $(15 + 100)/2 = 58 \text{ }^\circ\text{C}$ , and air properties can be estimated to be  $\rho = p/(RT) = 105/(287 \cdot 331) = 1.05 \text{ kg}/\text{m}^3$ ,  $c_p = 1010 \text{ J}/(\text{kg} \cdot \text{K})$ ,  $k = k_0(T/T_0)^{1/2} = 0.025(331/288)^{1/2} = 0.029 \text{ W}/(\text{m} \cdot \text{K})$ ,  $a = k/(\rho c_p) = 14 \cdot 10^{-6} \text{ m}^2/\text{s}$ ,  $\nu = \nu_0(T/T_0)^{1/2}/(\rho/\rho_0) = 17 \cdot 10^{-6} \text{ m}^2/\text{s}$ ,  $Pr = \nu/a = 0.72$ .
- Reynolds number.  $Re = \nu L/\nu = 10 \cdot 0.5/15 \cdot 10^{-5} = 0.33 \cdot 10^6$  if along the short side, and  $Re = \nu L/\nu = 10 \cdot 2/15 \cdot 10^{-5} = 1.3 \cdot 10^6$  if along the longest side.
- Nusselt correlation. For the short case,  $Re = 0.33 \cdot 10^6$  is smaller than the typical transition to turbulent boundary-layer ( $Re_{cr} = 0.5 \cdot 10^6$ ) and the whole plate can be assumed under laminar flow; thus, we use Chilton-Colburn analogy for  $Pr > 0.6$ ,  $Nu_L = 0.66 Pr^{1/3} Re_L^{1/2}$ , obtaining  $Nu = 0.66 \cdot 0.72^{1/3} \cdot (0.33 \cdot 10^6)^{1/2} = 340$ , and thence  $h = kNu/L = 340 \cdot 0.029/0.5 = 19.7 \text{ W}/(\text{m}^2 \cdot \text{K})$ , which means  $\dot{q} = h(T_w - T_\infty) = 1.67 \text{ kW}/\text{m}^2$  and  $\dot{Q} = \dot{q}A = 1.67 \cdot (0.5 \cdot 2) = 1.67 \text{ kW}$  in total.
- If the flow-along-the-longest-side case is considered,  $Re = 1.3 \cdot 10^6 > Re_{cr} = 0.5 \cdot 10^6$ ; thus, the flow was laminar for a while and then changed to turbulent, so we take for  $Pr > 0.6$   $Nu_L = 0.037 Pr^{1/3} Re_L^{4/5}$ , what gives  $Nu = 0.037 \cdot 0.72^{1/3} \cdot (1.3 \cdot 10^6)^{4/5} = 2580$ , and thence  $h = kNu/L = 2580 \cdot 0.029/2 = 37.4 \text{ W}/(\text{m}^2 \cdot \text{K})$ , which means  $\dot{q} = h(T_w - T_\infty) = 3.18 \text{ kW}/\text{m}^2$  and  $\dot{Q} = \dot{q}A = 3.18 \cdot (0.5 \cdot 2) = 3.18 \text{ kW}$  in total.
- Notice that blowing along the longest side significantly increases the rate of heat transfer (doubles it in our case).

## Temperature and pressure effects on fluid properties

In most heat convection correlations, fluid properties are evaluated at a fixed temperature dependent on the problem (the reference temperature), and assumed constant at every point and time. Two questions then arise: how to choose the best reference temperature, and how further corrections should be introduced, if any. Prandtl number,  $Pr = \nu/a$ , is entirely dependent on fluid properties, and enters into most heat convection correlations. Reynolds number,  $Re = \nu L/\nu$ , is dominated by viscosity. Thermal expansion governs free-convection parameters like  $Ra$  and  $Gr$ .

The reference temperatures commonly used are:

- For heat convection between a wall at  $T_w$  and a fluid with a far temperature  $T_\infty$ , the so-called film temperature  $T_f = (T_w + T_\infty)/2$  is used as reference to evaluate all fluid properties.

- For heat convection in a pipe, the most appropriate temperature reference is the inlet-and-outlet bulk average, although, when the outlet is unknown, the inlet mean temperature-difference between the fluid and pipe wall is used as reference.

From the several fluid properties involved in heat convection without phase change (density,  $\rho$ , viscosity,  $\mu$ , thermal conductivity,  $k$ , thermal capacity,  $c_p$ , thermal expansion coefficient,  $\alpha$ , and so on), viscosity of liquids is the most sensitive to temperature variations, all the others being mildly dependent on temperature and pressure.

### Gases

- Density. Density in gases can usually be computed from ideal gas law  $\rho=p/(RT)$  without any other correction. Temperature affects almost all gas properties. Pressure, if not below  $10^2$  Pa, has a negligible effect on  $c_p$ ,  $k$ , and  $\mu$  (and on  $Pr\equiv\mu c_p/k$ ), but the Reynolds number is almost proportional to gas pressure ( $Re\equiv vL/\nu \propto p$ ), and Rayleigh number,  $Ra\equiv\alpha g\Delta TL^3/(\nu\alpha)$ , proportional to  $p^2$  (and hence the convective coefficient in natural convection is proportional to the square root of pressure,  $h=h_0\sqrt{p/p_0}$ ). For  $p<10^2$  Pa the continuum model starts to fail, and for  $p<10^1$  Pa the molecular mean free path becomes comparable or larger than the size of the object of interest, and kinetic gas theory must be applied.
- Thermal capacity. Slowly increases with temperature (polynomial fittings are common); e.g. for air at 100 kPa,  $c_p=1032$  J/(kg·K) at 100 K and  $c_p=1141$  J/(kg·K) at 1000 K.
- Thermal conductivity and dynamic viscosity. These two transport coefficients increase with temperature with a small power-law ( $k=k_0(T/T_0)^n$  and  $\mu=\mu_0(T/T_0)^m$ ); although kinetic theory of gases show that both, thermal conductivity and dynamic viscosity, grow with  $T^{1/2}$ , a linear fit better fits the data. For more precise correlations, the following exponents have been proposed:  $n=0.71$  and  $m=0.65$  for monatomic gases,  $n=0.86$  and  $m=0.70$  for diatomic gases,  $n=1.3$  and  $m=0.88$  for triatomic gases. Both transport coefficients are nearly unchanged by pressure up to 10 MPa, increasing to a double value at some 50 MPa.
- Thermal diffusivity. A general dependence with temperature and pressure is of the form  $T^n/p$ , with  $1.5<n<2$  ( $n=3/2$  according to simple kinetic gas theory); e.g. for air at 100 kPa,  $a=2.54\cdot 10^{-6}$  m<sup>2</sup>/s at 100 K and  $a=168\cdot 10^{-6}$  m<sup>2</sup>/s at 1000 K.
- Prandtl number. Gases have a typical value of  $Pr=0.7$  and a small temperature dependence, with  $0.5<Pr<1$  under most  $p$ - $T$ -conditions; e.g. for air at 100 kPa,  $Pr=0.79$  at 100 K and  $Pr=0.73$  at 1000 K.

### Liquids

- Density. Temperature affects almost all liquid properties. The linear thermal expansion model for the density of liquids is good enough in most cases (far from the critical-point conditions). Saturated water, for instance, has  $\rho=1002$  kg/m<sup>3</sup> at 0 °C,  $\rho=960.6$  kg/m<sup>3</sup> at 100 °C and  $\rho=322$  kg/m<sup>3</sup> at its critical point, 374 °C. Pressure has a negligible effect on liquids (above their vapour pressure).
- Thermal capacity. May slightly increase or decrease with temperature; e.g. for saturated water,  $c_p=4218$  J/(kg·K) at 0 °C and  $c_p=5728$  J/(kg·K) at 300 °C.

- Viscosity. A simple temperature-corrections for liquid viscosity is:

$$\frac{\mu(T)}{\mu^{\oplus}} = \exp \left[ -C \left( 1 - \frac{T^{\oplus}}{T} \right) \right] \quad (42)$$

with a constant value of around  $C \approx 7$ ; e.g. for water,  $C=6$ , with  $\mu^{\oplus}=0.0011$  Pa·s at  $T^{\oplus}=288$  K; for many oils,  $C=8$ , with a typical value of (see [Liquid data](#))  $\mu^{\oplus}=0.1$  Pa·s at  $T^{\oplus}=288$  K, and so on. Saturated water, for instance, has  $\nu=1.79$  m<sup>2</sup>/s at 0 °C and  $\nu=0.135$  m<sup>2</sup>/s at 300 °C.

- Prandtl number. Most liquids, except viscous oils and liquid metals, have a range of  $2 < Pr < 20$  (e.g., at 15 °C,  $Pr=7$  for water,  $Pr=8$  for n-octane,  $Pr=19$  for ethanol), decreasing with temperature (e.g.  $Pr=1.02$  for saturated water at 300 °C,  $Pr=13.6$  at 0 °C). Viscous liquids like oils and glycerine may have large Prandtl-values,  $50 < Pr < 10^5$  (larger values are seldom considered as convecting), quickly decreasing when temperature is increased. Most liquids metals, under most  $p$ - $T$ -conditions, have  $0.005 < Pr < 0.05$ , with a typical value of  $Pr=0.01$ .

After having computed all fluid properties at the appropriate reference temperature, some corrections may be due, to account for the different possible temperature gradients. For gases, when the wall-to-bulk temperature ratios is in the range  $0.5 < T_w/T_{\infty} < 2$ , using the film temperature is good enough. For liquids, and for gases at high wall-to-bulk temperature ratios, heat convection correlations computed with mean film temperatures are modified with a viscosity factor, as the  $(\mu_w/\mu_{\infty})^{1/4}$  term in Table 5.

### [Forced and natural convection \(aside\)](#)

### [Convection with phase change \(aside\)](#)

### [Heat exchangers \(aside\)](#)

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